Basic Math - Third lesson

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1 Example of poly GCD computations

Example 1 (First Example)**.** *We want to determine the number of roots for* $f(x) = x^3 - 6x - 1$.

```
F:= x^3 - 6x - 1;
GCDPolySTURMVerbose(F,3x^2 - 6);


GCD(x^3 - 6x - 1,3x^2 - 6)=GCD(x^3 - 6x - 1,x^2 - 2)
DivPoly(x^3 - 6x - 1,x^2 - 2);

Passo 1 I have        x^3 - 6x - 1 moltiplico per x
Passo 1 I subtract x^3 - 2x
Passo 1 get       -4x - 1

Resto=-4x - 1
Quoto=x

(x^3 - 6x - 1)=(x)*(x^2 - 2)+(-4x - 1)

Change the remainder's sign

GCD(x^2 - 2,4x + 1)
DivPoly(x^2 - 2,4x + 1);

Passo 1 ho        x^2 - 2 moltiplico per 1/4x
Passo 1 sottraggo x^2 + 1/4x
Passo 1 ottengo       -1/4x - 2

Passo 2 ho        -1/4x - 2 moltiplico per -1/16
Passo 2 sottraggo -1/4x - 1/16
Passo 2 ottengo         -31/16

Resto=-31/16
Quoto=1/4x - 1/16
```

```
Change the remainder's sign

GCD(4x + 1,31/16)=GCD(4x + 1,1)=1

List of pairs  [[x^3 - 6x - 1, 3x^2 - 6], [x^2 - 2, 4x + 1], [4x + 1, 1]]
remainder Sequence = [x^3 - 6x - 1, 3x^2 - 6, 4x + 1, 1]

p=++++ 0
q=-+-+ 3
#roots=3
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```

Exercise 2 (Second Example)**.** *We want to determine the number of roots of* $f(x) = x^3 - 3x^2 + 3$.

```
F:= x^3-3x^2+3;
GCDPolySTURMVerbose(F,3x^2 - 6x);

GCD(x^3-3x^2+3,3x^2 - 6x)=GCD(x^3-3x^2+3,x^2 - 2x)
-- we can see the GCD will be 1, but we need the remainder sequence

DivPoly(x^3-3x^2+3,x^2 - 2x);

DivPoly(x^3-3x^2+3,x^2 - 2x);
Passo 1 ho        x^3 - 3x^2 + 3 moltiplico per x
Passo 1 sottraggo x^3 - 2x^2
Passo 1 ottengo       -x^2 + 3

Passo 2 I have        -x^2 + 3 moltiplico per -1
Passo 2 I subtract   -x^2 + 2x
Passo 2 I get        -2x + 3

Resto=-2x + 3
Quoto=x - 1

(x^3 - 3x^2 + 3)=(x - 1)*(x^2 - 2x)+(-2x + 3)
Change the remainder's sign

GCD(x^2 - 2x,2x - 3)
DivPoly(x^2 - 2x,2x - 3);

Passo 1 I have        x^2 - 2x moltiplico per 1/2x
Passo 1 I subtract x^2 - 3/2x
Passo 1 I get       -1/2x

Passo 2 I have        -1/2x moltiplico per -1/4
Passo 2 I subtract -1/2x + 3/4
```

```
Passo 2 I get          -3/4

Resto=-3/4
Quoto=1/2x - 1/4

(x^2 - 2x)=(1/2x - 1/4)*(2x - 3)+(-3/4)
Change the remainder's sign

GCD(2x - 3,x^2 - 2x,3/4) = GCD(2x - 3,x^2 - 2x,1)=1

(2x - 3)=(2x - 3)*(1)+(0)
List of pairs  [[x^3 - 3x^2 + 3, 3x^2 - 6x], [x^2 - 2x, 2x - 3], [2x - 3, 1]]
remainder Sequence = [x^3 - 3x^2 + 3, 3x^2 - 6x, 2x - 3, 1]
p=++++
q=+-+-
Numero radici 3
```

2 Root intervals

Proposition 3. *Let* $f(x) = a_d x^d + \cdots + a_1 x + a_0 \in \mathbb{R}[x]$. *Then if* $C = \max(\mid a_d \mid, \ldots, \mid a_0 \mid)$

$$\mid x_0 \mid > d\frac{C}{\mid a_d \mid} = M \Rightarrow f(x_0) \neq 0$$

This means that all the roots lie within interval

$$\left(-\frac{Cd}{\mid a_d \mid}, \frac{Cd}{\mid a_d \mid}\right)$$

Example 4. *Consider the polynomial* $f(x) = 5x^5 + 2x^3 - 6$. *Here* $d = 5$ *and*

$$C = \max(\mid 5 \mid, \mid 2 \mid, \mid -6 \mid) = 6$$

Then the real roots of the polynomial $f(x)$ *lie in the interval*

$$\left(-5 \cdot \frac{6}{5}, 5 \cdot \frac{6}{5}\right) = (-6, 6)$$

Example 5. *Let* $f(x) = 7x^6 - 12x^3 - 5$ *and hence* $d = 6$ *and* $C = \max(\mid 7 \mid, \mid -12 \mid, \mid -5 \mid) = 12$. *Then the real roots of the polynomial* $f(x)$ *lie in the interval*

$$\left(-6 \cdot \frac{12}{7}, 6 \cdot \frac{12}{7}\right) = \left(-\frac{72}{7}, \frac{72}{7}\right)$$

Since $\frac{72}{7} \simeq 10.28$, *to simplify the computations, we can choose a slighty larger interval,* $(-11, 11)$.

Proposition 6 (Sturm Algorithm for intervals)**.** *Let* $f(x) \in \mathbb{R}[x]$, *and* $f(x), f'(x), r_1(x), \ldots, r_n(x)$ *the sequence of remainders produced by the Sturm algorithm. Let* $a, b \in \mathbb{R}$. *Let*

- *p be the number of sign variation in the sequence $f(a), f'(a), r_1(a), \ldots, r_n(a)$.*
- *q be the number of sign variation in the sequence $f(b), f'(b), r_1(b), \ldots, r_n(b)$.*

Then the number of roots of $f(x)$ in the interval (a, b) is $\mid p - q \mid$.

Remark 7. *When we compute the number of sign variations in a sequence, we skip a 0 as non-existent. For example*

- *the number of sign variations of $[+ + 0 + +]$ is 0;*
- *the number of sign variations of $[+ + 0 - +]$ is 2;*

Remark 8. *We want to determine the number of roots and, for every root x_0, an interval (a, b) in which x_0 is the only root. We compute the number of roots with the Sturm Algorithm. As a byproduct, we have the remainder sequence. We then use the remainder sequence to determine an interval for every root by splitting the interval $(-M, M)$.*

Remark 9. *Let $f(x) = \in \mathbb{R}[x]$. With the notation of Proposition 3 above,*

- *the signs of $f(M), f'(M), r_1(M), \ldots, r_n(M)$ are the signs of the leading coefficients of $f(x), f'(x), r_1(x), \ldots, r_n(x)$*
- *the signs of $f(-M), f'(-M), r_1(-M), \ldots, r_n(-M)$ are the signs of the leading coefficients of $f(x), f'(x), r_1(x), \ldots, r_n(x)$ changing the sign for the odd degree polynomials.*

Example 10 (First Example)**.** *We want to determine the numer of roots and the root intervals for $f(x) = x^3 - 6x - 1$. The details of the STURM Algorithm computations are shown in Example 1.*

```
F:= x^3 - 6x - 1;
GCDPolySTURMVerbose(F,Der(F,x));

(x^3 - 6x - 1)=(x)*(x^2 - 2)+(-4x - 1)
(x^2 - 2)=(1/4x - 1/16)*(4x + 1)+(-31/16)
(4x + 1)=(4x + 1)*(1)+(0)
Lista delle coppie  [[x^3 - 6x - 1, 3x^2 - 6], [x^2 - 2, 4x + 1], [4x + 1, 1]]
Record[GCD = 1, Sequence = [x^3 - 6x - 1, 3x^2 - 6, 4x + 1, 1]]
-------------------------------
```

The 3 roots are in the interval $(-18, 18)$ To separate the roots, we find the number of roots in $(-18, 0)$ and $(0, 18)$. The remainder sequence is $f_1(x) = x^3-6x-1, f_2(x) = 3x^2-6, f_3(x) = 4x+1, f_4(x) = 1$ We split $(-18, 18)$

- $(-18, 0)$. *We have*

  $$f_1(-18) = -5725, f_2(-18) = 966, f_3(-18) = -71, f_4(-18) = 1 \quad \Rightarrow p = 3$$

  and

  $$f_1(0) = -1, f_2(0) = -6, f_3(0) = 1, f_4(0) = 1 \quad \Rightarrow q = 1$$

  and the number of roots in $(-18, 0)$ is $p - q = 2$.

- *Since there are 3 roots,* $\boxed{\textit{the number of roots in } (0,18) \textit{ is } 1}$. *OK*

We split $(-18,0)$

- $(-18,-9)$. *We have*

$$f_1(-18) = -5725, f_2(-18) = 966, f_3(-18) = -71, f_4(-18) = 1 \quad \Rightarrow p_{-18} = 3$$

  and

$$f_1(-9) = -676, f_2(-9) = 237, f_3(-9) = -35, f_4(-9) = 1 \quad \Rightarrow q_{-9} = 3$$

  and the number of roots in $(-18,0)$ *is* $p - q = 0$.

- *Since there are* 2 *roots in* $(-18,0)$*, the number of roots in* $(-9,0)$ *is* 2*, and .*

We split $(-18,0)$

- $(-9,0)$. *We have*

$$f_1(-9) = -676, f_2(-9) = 237, f_3(-9) = -35, f_4(-9) = 1 \quad \Rightarrow q_{-9} = 3$$

  and

$$f_1(-4) = -676, f_2(-4) = 42, f_3(-4) = -15, f_4(-4) = 1 \quad \Rightarrow q_{-9} = 3$$

  and the number of roots in $(-9,4)$ *is* $p - q = 0$.

- *Since there are* 2 *roots in* $(-9,0)$*, the number of roots in* $(-4,0)$ *is* 2.

We split $(-18,0)$

- $(-4,-2)$. *We have*

$$f_1(-4) = -676, f_2(-4) = 42, f_3(-4) = -15, f_4(-4) = 1 \quad \Rightarrow q_{-9} = 3$$

  and

$$f_1(-2) = 3, f_2(-2) = 6, f_3(-2) = -7, f_4(-2) = 1 \quad \Rightarrow q_{-9} = 2$$

  and $\boxed{\textit{the number of roots in } (-4,2) \textit{ is } p - q = 1}$.

- *Since there are* 2 *roots in* $(-4,0)$, $\boxed{\textit{the number of roots in } (-2,0) \textit{ is } 1}$.

So we have one root each in the intervals $(-4,2)$, $(-2,0)$, $(0,18)$.

3 Homework

Exercise 11 (Second Example)**.** *We want to determine the numer of roots and the root intervals for* $P_1(x) = x^3 - 6x - 1$. *The details of the STURM Algorithm computations are shown in Example 2.*

```
F:= x^3-3x^2+3;
GCDPolySTURMVerbose(F,Der(F,x));

(x^3 - 3x^2 + 3)=(x - 1)*(x^2 - 2x)+(-2x + 3)
(x^2 - 2x)=(1/2x - 1/4)*(2x - 3)+(-3/4)
(2x - 3)=(2x - 3)*(1)+(0)
Lista delle coppie  [[x^3 - 3x^2 + 3, 3x^2 - 6x], [x^2 - 2x, 2x - 3], [2x - 3, 1]]
Record[GCD = 1, Sequence = [x^3 - 3x^2 + 3, 3x^2 - 6x, 2x - 3, 1]]
p=++++ 0
q=-+-+ 3
Numero radici 3

Big Interval(-3*3,3*3)=(-9,9)


Subst(V,x,-9); Equal to the STURM p sequence  3
Subst(V,x,0); [3, 0, -3, 1] ++-+ 2
In (-9,0) one root, **************** INTERVAL FOUND
so in (0,9) two roots

In (0,4) two roots
Subst(V,x,0); [3, 0, -3, 1] +0-+ 2
Subst(V,x,4); [19, 24, 5, 1] ++++ 0

In (0,2) one root
Subst(V,x,0); [3, 0, -3, 1] +0-+ 2
Subst(V,x,2); [-1, 0, 1, 1] -+++ 1

So one root  in (2,4)  ********INTERVAL FOUND
and the other root in (0,2) *****INTERVAL FOUND
```

Example 12. *We want to determine the root intervals for* $P_2(x) = 2x^3 - 6x - 3$.

```
F:= 2x^3-6x-3;
GCDPolySTURMVerbose(F,Der(F,x));

(2x^3 - 6x - 3)=(2x)*(x^2 - 1)+(-4x - 3)
(x^2 - 1)=(1/4x - 3/16)*(4x + 3)+(-7/16)
(4x + 3)=(4x + 3)*(1)+(0)
Lista delle coppie  [[2x^3 - 6x - 3, 6x^2 - 6], [x^2 - 1, 4x + 3], [4x + 3, 1]]
remainder Sequence = [2x^3 - 6x - 3, 6x^2 - 6, 4x + 3, 1]
```

Solutions: 3 *real roots,* $x \simeq -1.38, -0.55, 1.94$.

Exercise 13. *We want to determine the root intervals for* $P_3(x) = x^3 - 3x - 1$.

```
F:= x^3-3x-1;
GCDPolySTURMVerbose(F,Der(F,x));

(x^3 - 3x - 1)=(x)*(x^2 - 1)+(-2x - 1)
(x^2 - 1)=(1/2x - 1/4)*(2x + 1)+(-3/4)
(2x + 1)=(2x + 1)*(1)+(0)
Lista delle coppie  [[x^3 - 3x - 1, 3x^2 - 3], [x^2 - 1, 2x + 1], [2x + 1, 1]]
remainder Sequence = [x^3 - 3x - 1, 3x^2 - 3, 2x + 1, 1]
```

Solutions: 3 real roots, $x \simeq -1.53, -0.34, 1.87$.

Exercise 14. *We want to determine the root intervals for* $P_4(x) = x^5 - 3x^4 + 1$.

```
F:= x^5-3x^4+1;
GCDPolySTURMVerbose(F,Der(F,x));

(x^5 - 3x^4 + 1)=(1/5x - 3/25)*(5x^4 - 12x^3)+(-36/25x^3 + 1)
(5x^4 - 12x^3)=(5/36x - 1/3)*(36x^3 - 25)+(125/36x - 25/3)
(36x^3 - 25)=(-36/5x^2 - 432/25x - 5184/125)*(-5x + 12)+(59083/125)
(-5x + 12)=(5x - 12)*(-1)+(0)
Lista delle coppie  [[x^5 - 3x^4 + 1, 5x^4 - 12x^3], [5x^4 - 12x^3, 36x^3 - 25], [36x^3 - 25, -5x
remainder Sequence = [x^5 - 3x^4 + 1, 5x^4 - 12x^3, 36x^3 - 25, -5x + 12, -1]
```

Solutions: 3 real roots, $x \simeq -0.72, 0.82, 2.98$.

Exercise 15. *We want to determine the root intervals for* $P_5(x) = x^4 - x^3 - 1$.

```
F:= x^4-x^3-1;
GCDPolySTURMVerbose(F,Der(F,x));

(x^4 - x^3 - 1)=(1/4x - 1/16)*(4x^3 - 3x^2)+(-3/16x^2 - 1)
(4x^3 - 3x^2)=(4/3x - 1)*(3x^2 + 16)+(-64/3x + 16)
(3x^2 + 16)=(3/4x + 9/16)*(4x - 3)+(283/16)
(4x - 3)=(-4x + 3)*(-1)+(0)
Lista delle coppie  [[x^4 - x^3 - 1, 4x^3 - 3x^2], [4x^3 - 3x^2, 3x^2 + 16], [3x^2 + 16, 4x - 3],
remainder Sequence = [x^4 - x^3 - 1, 4x^3 - 3x^2, 3x^2 + 16, 4x - 3, -1]
```

Solutions: 2 real roots, $x \simeq -0.81, 1.38$.