Basic Math - First lesson

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1 Notations

1. The set of natural numbers (positive integers) is denoted by $\mathbb{N}$. The elements of $\mathbb{N}$ are $0, 1, 2, \cdots$ etc..
2. The set of integer numbers is denoted by $\mathbb{Z}$. The elements of $\mathbb{Z}$ are $0, 1, -1, -2, -2 \cdots$ etc..
3. The set of rationals (numeric fractions) is denoted by $\mathbb{Q}$. The elements of $\mathbb{Q}$ are $-3, 0, 2, \frac{4}{5}$ etc..
4. The set of reals is denoted by $\mathbb{R}$. Elements of R are $-3, 0, 2, \frac{4}{5}, \sqrt{2}, \pi$ etc..
5. Note that all these sets are nested $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$
6. The set of polynomials over $\mathbb{R}$ (with real coefficients) is denoted by $\mathbb{R}[x]$. Elements of $\mathbb{R}[x]$ are $\pi x + 1, x^3 - \frac{2}{5}x^2 + 1, x^2 - \sqrt{2}$ etc..
7. The sets $\mathbb{N}[x], \mathbb{Z}[x], \mathbb{Q}[x]$ are defined similarly.
8. Note that these sets are nested $\mathbb{N}[x] \subset \mathbb{Z}[x] \subset \mathbb{Q}[x] \subset \mathbb{R}[x]$.

Definition 1. *Let $n \in \mathbb{N}$. The set of divisors of n, denoted* $\mathrm{DIV}(n)$*, is the set of all natural numbers that divide n. For example,*

$$\mathrm{DIV}(12) = \{1, 2, 3, 4, 6, 12\}.$$

2 Theorems and Propositions

Proposition 2. *For all $a, b \in \mathbb{R}[x]$, the following hold:*

- $a^2 - b^2 = (a + b)(a - b)$.
- $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.
- $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.
- $a^2 + b^2$ *cannot be factored further over* $\mathbb{R}$.

Theorem 3 (Ruffini's Theorem)**.** *Let $f(x) \in \mathbb{R}[x]$ and $a \in \mathbb{R}$ such that $f(a) = 0$ (i.e., a is a root of $f(x)$). Then,*

$$(x - a) \mid f(x).$$

Proposition 4. *Let $f(x), g(x), h(x) \in \mathbb{R}[x]$ such that $f(x) \mid h(x)$, $g(x) \mid h(x)$, and $f(x), g(x)$ are coprime. Then,*

$$f(x)g(x) \mid h(x).$$

Theorem 5 (Rational Root Theorem)**.** *Let $f(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathbb{Z}[x]$ and $a \in \mathbb{Q}$ such that $f(a) = 0$. Then,*

$$a \in \left\{ \pm\frac{p}{q} \mid p \in \mathrm{DIV}(a_0),\ q \in \mathrm{DIV}(a_d) \right\}.$$

3 Exercises

1. $P(x) = x^5 + x^3 + x^2 + 1 = 0$. Note that $P(-1) = 0$. Use polynomial division.
   - Factorization $x^5 + x^3 + x^2 + 1 = (x^2 + 1)(x^3 + 1) = (x + 1)(x^2 + 1)(x^2 - x + 1)$.
   - The solution is $x = -1$.
2. $P(x) = x^5 - 9x^3 - 8x^2 + 72 = 0$. Note that $P(\pm 3) = P(2) = 0$. Use division.
   - Factorization $x^5 - 9x^3 - 8x^2 + 72 = (x + 3)(x - 3)(x - 2)(x^2 + 2x + 4)$.
   - The solutions are $x = \pm 3, 2$.
3. $P(x) = x^4 - 3x^3 + 2x - 6 = 0$. Note that $P(3) = 0$. Use division.
   - Factorization $x^4 - 3x^3 + 2x - 6 = (x - 3)(x^3 + 2)$.
   - The solutions are $x = 3,\ \pm\sqrt[3]{-2}$.
4. For the parameter $a \in \mathbb{R}$, $P(x) = x^3 - ax^2 - 2x + 2a = 0$. Note that $P(a) = 0$. Use division.
   - Factorization $x^3 - ax^2 - 2x + 2a = (x - a)(x^2 - 2)$.
   - The solutions are $x = a,\ \pm\sqrt{2}$.
5. For the parameter $a \in \mathbb{R}$, $P(x) = x^3 - ax^2 - 2x + 2a = 0$. Note that $P(a) = 0$. Use division.
   - Factorization $x^3 - ax^2 - ax + a^2 = (x - a)(x^2 - a)$.
   - The solutions are $x = a$ always, if $a \geq 0$ also $\pm\sqrt{a}$.

4 Proposed exercises

Solutions will be given in the next installment of these notes. Regrouping will be more difficult here. Use of the root rule and Ruffini is recommended.

1. $x^3 2 - 1 = 0$
2. $x^8 - 4 = 0$
3. $x^3 + (1 - a)x^2 - (a + 6)x + 6a = 0$
4. $2x^3 - 17x^2 + 38x - 15 = 0$
5. $3x^4 - 22x^3 - 2x^2 + 66x - 21 = 0$
6. $x^3 - 17x^2 + 92x - 160 = 0$
7. $x^6 - 12x^4 + 47x^2 - 60 = 0$
8. $x^3 - xa^2 - 2xab - xb^2 - x^2 + a^2 + 2ab + b^2 = 0$
9. $x^3 + 3x^2y + 3xy^2 + y^3 - 2x^2 - 4xy - 2y^2 - x - y + 2 = 0$. Hint: try to detect powers.