# Classwork 13_F

Solutions

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Each exercise is marked with its level. In the solutions $k$ and $h$ denote arbitrary integers.

**Exercise 1** (Polynomial gcd). **Level: medium**

*Compute, with the Euclidean algorithm, the gcd of*

$$A(x) = x^6 - 1 \qquad \textit{and} \qquad B(x) = x^4 - 1$$

First division, $A$ by $B$:

$$\begin{array}{rr|l} x^6 & -1 & x^4-1 \\ \cline{3-3} -x^6+x^2 & & x^2 \\ \cline{1-1} x^2-1 & & \end{array}$$

The remainder is $R_1(x) = x^2 - 1$. Second division, $B$ by $R_1$:

$$\begin{array}{rr|l} x^4 & -1 & x^2-1 \\ \cline{3-3} -x^4+x^2 & & x^2+1 \\ \cline{1-1} x^2-1 & & \\ -x^2+1 & & \\ \cline{1-1} 0 & & \end{array}$$

The remainder is 0, so the gcd is $x^2 - 1$.

Remark: $x^6 - 1 = (x^2 - 1)(x^4 + x^2 + 1)$ and $x^4 - 1 = (x^2 - 1)(x^2 + 1)$, and $x^4 + x^2 + 1$ and $x^2 + 1$ have no common factor. In general $\gcd(x^m - 1, x^n - 1) = x^{\gcd(m,n)} - 1$.

**Answer:** $\gcd(A, B) = x^2 - 1$.

**Exercise 2** (Parabolas). **Level: medium**

*Find the vertex of the parabola $y = x^2 - 2x - 3$ and its intersections with the line $y = x + 1$.*

Vertex: $x_V = 1$, $y_V = 1 - 2 - 3 = -4$, so $V = (1, -4)$.

Intersections: $x^2 - 2x - 3 = x + 1 \iff x^2 - 3x - 4 = 0 \iff (x - 4)(x + 1) = 0$. For $x = 4$, $y = 5$; for $x = -1$, $y = 0$.

**Answer:** $V = (1, -4)$; intersections $(4, 5)$ and $(-1, 0)$.

**Exercise 3** (Circles). **Level: difficult**

*Write the equations of the two tangent lines to the circle $x^2 + y^2 = 25$ passing through the point $P = (7, 1)$.*

$P$ is outside the circle, since $49 + 1 > 25$. The non-vertical lines through $P$ are $y - 1 = m(x - 7)$, that is $mx - y + 1 - 7m = 0$ (the vertical line $x = 7$ does not meet the circle). A line is tangent when its distance from the center $O = (0, 0)$ equals the radius 5:

$$\frac{|1 - 7m|}{\sqrt{m^2 + 1}} = 5 \iff (1 - 7m)^2 = 25(m^2 + 1) \iff 24m^2 - 14m - 24 = 0 \iff 12m^2 - 7m - 12 = 0$$

so $m = \frac{7 \pm \sqrt{49 + 576}}{24} = \frac{7 \pm 25}{24}$, that is $m = \frac{4}{3}$ or $m = -\frac{3}{4}$.

**Answer:** $4x - 3y - 25 = 0$ and $3x + 4y - 25 = 0$.

**Exercise 4** (Exponential equations). **Level: difficult**

*Solve* $4^{x+1} - 9 \cdot 2^x + 2 = 0$.

Since $4^{x+1} = 4 \cdot (2^x)^2$, with $t = 2^x > 0$:

$$4t^2 - 9t + 2 = 0 \iff t = \frac{9 \pm 7}{8} \iff t = 2 \text{ or } t = \frac{1}{4}$$

Both positive. $2^x = 2 \iff x = 1$ and $2^x = \frac{1}{4} = 2^{-2} \iff x = -2$.

**Answer:** $x = 1$ or $x = -2$.

**Exercise 5** (Logarithmic equations). **Level: difficult**

*Solve* $\log_2(x-1) = \log_4(x+5)$.

Domain: $x - 1 > 0$ and $x + 5 > 0$, that is $x > 1$. Since $\log_4(x+5) = \frac{\log_2(x+5)}{\log_2 4} = \frac{1}{2}\log_2(x+5)$,

$$2\log_2(x-1) = \log_2(x+5) \iff \log_2(x-1)^2 = \log_2(x+5) \iff (x-1)^2 = x+5 \iff x^2-3x-4 = 0$$

so $x = 4$ or $x = -1$, and only $x = 4$ is in the domain. Check: $\log_2 3$ and $\log_4 9 = \log_2 3$.

**Answer:** $x = 4$.

**Exercise 6** (Trigonometry). **Level: difficult**

*Compute* $\cos\frac{\pi}{8}$, $\sin\frac{\pi}{8}$ *and* $\tan\frac{\pi}{8}$ *exactly.*

From the duplication formulas, $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ and $\sin^2\theta = \frac{1-\cos 2\theta}{2}$. With $\theta = \frac{\pi}{8}$, $2\theta = \frac{\pi}{4}$:

$$\cos^2\frac{\pi}{8} = \frac{1 + \frac{\sqrt{2}}{2}}{2} = \frac{2+\sqrt{2}}{4}, \qquad \sin^2\frac{\pi}{8} = \frac{2-\sqrt{2}}{4}$$

Since $\frac{\pi}{8}$ is in the first quadrant, both are positive:

$$\cos\frac{\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2}, \qquad \sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}$$

For the tangent, $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{2\sin^2\theta}{2\sin\theta\cos\theta} = \frac{1-\cos 2\theta}{\sin 2\theta}$, so

$$\tan\frac{\pi}{8} = \frac{1 - \frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \frac{2-\sqrt{2}}{\sqrt{2}} = \sqrt{2} - 1$$

**Exercise 7** (Trigonometric equations). **Level: medium**

*Solve* $\sin\left(x + \frac{\pi}{3}\right) = \cos x$.

Since $\cos x = \sin\left(\frac{\pi}{2} - x\right)$, the equation is $\sin\left(x + \frac{\pi}{3}\right) = \sin\left(\frac{\pi}{2} - x\right)$:

- $x + \frac{\pi}{3} = \frac{\pi}{2} - x + 2k\pi \iff 2x = \frac{\pi}{6} + 2k\pi \iff x = \frac{\pi}{12} + k\pi$;
- $x + \frac{\pi}{3} = \pi - \left(\frac{\pi}{2} - x\right) + 2k\pi = \frac{\pi}{2} + x + 2k\pi \iff \frac{\pi}{3} = \frac{\pi}{2} + 2k\pi$, which is impossible.

**Answer:** $x = \frac{\pi}{12} + k\pi$.

**Exercise 8** (Trigonometric equations). **Level: difficult**

*Solve* $\sin x - \sqrt{3}\cos x = \sqrt{3}$ *using the tangent of the half angle.*

Here $a = 1$, $b = -\sqrt{3}$, $c = \sqrt{3}$.

1. For $x = \pi$: $0 - \sqrt{3}\cdot(-1) = \sqrt{3}$. So $x = \pi + 2k\pi$ are solutions (indeed $b + c = 0$).
2. For $x \neq \pi + 2k\pi$, with $t = \tan\frac{x}{2}$:

$$\frac{2t}{1+t^2} - \sqrt{3}\,\frac{1-t^2}{1+t^2} = \sqrt{3} \iff 2t - \sqrt{3} + \sqrt{3}\,t^2 = \sqrt{3} + \sqrt{3}\,t^2 \iff t = \sqrt{3}$$

3. $\tan\frac{x}{2} = \sqrt{3} \iff \frac{x}{2} = \frac{\pi}{3} + k\pi \iff x = \frac{2\pi}{3} + 2k\pi$.

Check for $x = \frac{2\pi}{3}$: $\frac{\sqrt{3}}{2} - \sqrt{3}\cdot\left(-\frac{1}{2}\right) = \sqrt{3}$.
**Answer:** $x = \frac{2\pi}{3} + 2k\pi$ or $x = \pi + 2k\pi$.

**Exercise 9** (Logarithms and trigonometry)**.** **Level: difficult**

*Solve* $\log_3(\sin x) - \log_3(\cos x) = \frac{1}{2}$.

*Domain.* $\sin x > 0$ and $\cos x > 0$: $x$ in the first quadrant, $2k\pi < x < \frac{\pi}{2} + 2k\pi$.

In the domain, $\log_3 a - \log_3 b = \log_3 \frac{a}{b}$, so

$$\log_3(\tan x) = \frac{1}{2} \iff \tan x = 3^{1/2} = \sqrt{3} \iff x = \frac{\pi}{3} + h\pi$$

For $h$ even the angle is in the first quadrant and is accepted; for $h$ odd it is in the third quadrant, where the logarithms are not defined.

**Answer:** $x = \frac{\pi}{3} + 2k\pi$.