Classwork 13_E

Solutions

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Each exercise is marked with its level. In the solutions k and h denote arbitrary integers.

Exercise 1 (Polynomial gcd). **Level: medium**

Compute, with the Euclidean algorithm, the gcd of

$$A(x) = x^3 + 2x^2 - x - 2 \qquad \textit{and} \qquad B(x) = x^3 + 3x^2 + 2x$$

First division, A by B:

$$\begin{array}{rl} x^3 + 2x^2 \ - x - 2 & \big| \, x^3 + 3x^2 + 2x \\ \underline{- x^3 - 3x^2 - 2x} & \big| \, 1 \\ - x^2 - 3x - 2 & \end{array}$$

The remainder is $R_1(x) = -x^2 - 3x - 2$. Second division, B by R_1:

$$\begin{array}{rl} x^3 + 3x^2 + 2x & \big| \, - x^2 - 3x - 2 \\ \underline{- x^3 - 3x^2 - 2x} & \big| \, - x \\ 0 & \end{array}$$

The remainder is 0, so the monic gcd is $x^2 + 3x + 2 = (x + 1)(x + 2)$.

Check by factoring: $A(x) = x^2(x+2)-(x+2) = (x-1)(x+1)(x+2)$ and $B(x) = x(x+1)(x+2)$.

Answer: $\gcd(A, B) = x^2 + 3x + 2$.

Exercise 2 (Parabolas). **Level: difficult**

Find the parabola with vertical axis, vertex $V = (2, -1)$, passing through $P = (0, 3)$. Then find its intersections with the line $y = x - 1$.

A parabola with vertical axis and vertex $(2, -1)$ is $y = a(x - 2)^2 - 1$. Imposing P: $3 = 4a - 1$, so $a = 1$ and

$$y = (x - 2)^2 - 1 = x^2 - 4x + 3$$

Intersections with $y = x - 1$: $x^2 - 4x + 3 = x - 1 \iff x^2 - 5x + 4 = 0 \iff (x - 1)(x - 4) = 0$. For $x = 1$, $y = 0$; for $x = 4$, $y = 3$.

Answer: $y = x^2 - 4x + 3$; intersections $(1, 0)$ and $(4, 3)$.

Exercise 3 (Circles). **Level: difficult**

Find the common points of the circles $\Gamma_1 \colon x^2 + y^2 = 25$ and $\Gamma_2 \colon x^2 + y^2 - 14x + 17 = 0$. Give center and radius of Γ_2.

Completing the square, Γ_2 is $(x - 7)^2 + y^2 = 32$: center $(7, 0)$, radius $\sqrt{32} = 4\sqrt{2}$.

Subtracting the two equations we eliminate $x^2 + y^2$:

$$(x^2 + y^2) - (x^2 + y^2 - 14x + 17) = 25 - 0 \iff 14x - 17 = 25 \iff x = 3$$

(the line $x = 3$ is the *radical axis* of the two circles). Substituting in Γ_1: $9 + y^2 = 25$, so $y = \pm 4$. Check in Γ_2: $9 + 16 - 42 + 17 = 0$.

Answer: $(3, 4)$ and $(3, -4)$.

Exercise 4 (Exponential equations). **Level: medium**

Solve $2^{x+1} + 2^{x+3} = 80$.

Collecting 2^x: $2^x(2+8) = 80$, so $2^x = 8 = 2^3$ and $x = 3$. Check: $2^4 + 2^6 = 16 + 64 = 80$.

Answer: $x = 3$.

Exercise 5 (Logarithmic equations). **Level: difficult**

Solve $\log_2 x + \log_x 2 = \frac{5}{2}$.

Domain: $x > 0$ and $x \neq 1$ (the base of a logarithm must be positive and different from 1). By the change of base formula $\log_x 2 = \frac{1}{\log_2 x}$; with $L = \log_2 x \neq 0$:

$$L + \frac{1}{L} = \frac{5}{2} \iff 2L^2 - 5L + 2 = 0 \iff L = \frac{5 \pm 3}{4} \iff L = 2 \text{ or } L = \frac{1}{2}$$

$\log_2 x = 2 \iff x = 4$, and $\log_2 x = \frac{1}{2} \iff x = \sqrt{2}$. Both are in the domain.

Answer: $x = 4$ or $x = \sqrt{2}$.

Exercise 6 (Trigonometry). **Level: difficult**

Prove that $\dfrac{\sin 3x}{\sin x} - \dfrac{\cos 3x}{\cos x} = 2$ *for every* x *with* $\sin x \neq 0$ *and* $\cos x \neq 0$.

Writing the left side with a common denominator, and using the subtraction formula for the sine,

$$\frac{\sin 3x \cos x - \cos 3x \sin x}{\sin x \cos x} = \frac{\sin(3x - x)}{\sin x \cos x} = \frac{\sin 2x}{\sin x \cos x} = \frac{2 \sin x \cos x}{\sin x \cos x} = 2$$

We could simplify because $\sin x \cos x \neq 0$ by hypothesis.

Exercise 7 (Trigonometric equations). **Level: medium**

Solve $\sqrt{3} \tan x - 2 \sin x = 0$.

Domain: $x \neq \frac{\pi}{2} + k\pi$. Since $\tan x = \frac{\sin x}{\cos x}$,

$$\sin x \left(\frac{\sqrt{3}}{\cos x} - 2 \right) = 0$$

- $\sin x = 0 \iff x = k\pi$, which is in the domain;
- $\frac{\sqrt{3}}{\cos x} = 2 \iff \cos x = \frac{\sqrt{3}}{2} \iff x = \pm\frac{\pi}{6} + 2k\pi$.

Answer: $x = k\pi$ or $x = \pm\frac{\pi}{6} + 2k\pi$.

Exercise 8 (Trigonometric equations). **Level: difficult**

Solve $\sqrt{3} \cos x - \sin x = \sqrt{3}$ *using the tangent of the half angle.*

Writing it as $-\sin x + \sqrt{3} \cos x = \sqrt{3}$: $a = -1$, $b = \sqrt{3}$, $c = \sqrt{3}$.

1. For $x = \pi$: $-\sqrt{3} \neq \sqrt{3}$. The values $x = \pi + 2k\pi$ are not solutions.
2. For $x \neq \pi + 2k\pi$, with $t = \tan \frac{x}{2}$:

$$\sqrt{3} \frac{1 - t^2}{1 + t^2} - \frac{2t}{1 + t^2} = \sqrt{3} \iff \sqrt{3} - \sqrt{3}\, t^2 - 2t = \sqrt{3} + \sqrt{3}\, t^2$$
$$\iff 2\sqrt{3}\, t^2 + 2t = 0 \iff 2t \left(\sqrt{3}\, t + 1 \right) = 0$$

   so $t = 0$ or $t = -\frac{\sqrt{3}}{3}$.

3. $\tan \frac{x}{2} = 0 \iff x = 2k\pi$; $\quad \tan \frac{x}{2} = -\frac{\sqrt{3}}{3} \iff \frac{x}{2} = -\frac{\pi}{6} + k\pi \iff x = -\frac{\pi}{3} + 2k\pi$.

Check for $x = -\frac{\pi}{3}$: $\sqrt{3} \cdot \frac{1}{2} + \frac{\sqrt{3}}{2} = \sqrt{3}$.
Answer: $x = 2k\pi$ or $x = -\frac{\pi}{3} + 2k\pi$.

Exercise 9 (Trigonometric equations)**.** **Level: difficult**
Solve $\sin^6 x + \cos^6 x = \frac{5}{8}$.
Using $a^3 + b^3 = (a+b)^3 - 3ab(a+b)$ with $a = \sin^2 x$, $b = \cos^2 x$, $a + b = 1$:

$$\sin^6 x + \cos^6 x = 1 - 3\sin^2 x \cos^2 x = 1 - \frac{3}{4}\sin^2 2x$$

since $\sin x \cos x = \frac{1}{2}\sin 2x$. The equation becomes

$$1 - \frac{3}{4}\sin^2 2x = \frac{5}{8} \iff \sin^2 2x = \frac{1}{2} \iff \sin 2x = \pm\frac{\sqrt{2}}{2}$$

The solutions of $\sin 2x = \pm\frac{\sqrt{2}}{2}$ are $2x = \frac{\pi}{4} + \frac{k\pi}{2}$ (the four angles $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$, $\frac{7\pi}{4}$, plus $2k\pi$).
Answer: $x = \frac{\pi}{8} + \frac{k\pi}{4}$.