Classwork 13_D

Solutions

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Each exercise is marked with its level. In the solutions k and h denote arbitrary integers.

Exercise 1 (Polynomial gcd). **Level: difficult**

Compute, with the Euclidean algorithm, the gcd of

$$A(x) = x^4 + 2x^3 - 2x - 1 \qquad \textit{and} \qquad B(x) = x^4 - 2x^2 + 1$$

and then factor both polynomials.

First division, A by B:

$$\begin{array}{rrrr|l} x^4 & +2x^3 & -2x & -1 & x^4-2x^2+1 \\ \hline -x^4 & +2x^2 & & -1 & 1 \\ \hline & 2x^3 & +2x^2 & -2x-2 & \end{array}$$

The remainder is $R_1(x) = 2x^3 + 2x^2 - 2x - 2$. Second division, B by R_1:

$$\begin{array}{rrrr|l} x^4 & -2x^2 & & +1 & 2x^3+2x^2-2x-2 \\ \hline -x^4 & -x^3 & +x^2+x & & \frac{1}{2}x-\frac{1}{2} \\ \hline & -x^3 & -x^2+x & +1 & \\ & x^3 & +x^2-x & -1 & \\ \hline & & & 0 & \end{array}$$

The remainder is 0, so the monic gcd is $D(x) = x^3 + x^2 - x - 1$. Factoring it by grouping, $D(x) = x^2(x+1) - (x+1) = (x-1)(x+1)^2$. Then, from the divisions,

$$\begin{aligned} B(x) &= (x^2-1)^2 = (x-1)^2(x+1)^2 \\ A(x) &= B(x) + 2D(x) = (x-1)(x+1)^2\big((x-1)+2\big) = (x-1)(x+1)^3 \end{aligned}$$

Answer: $\gcd(A, B) = x^3 + x^2 - x - 1 = (x-1)(x+1)^2$.

Exercise 2 (Parabolas). **Level: difficult**

Find the value of k for which the line $y = 2x + k$ is tangent to the parabola $y = x^2 - 4x + 7$, and find the point of tangency.

The intersections are given by $x^2 - 4x + 7 = 2x + k$, that is

$$x^2 - 6x + 7 - k = 0$$

The line is tangent when this equation has a double root, that is when the discriminant is zero:

$$\frac{\Delta}{4} = 9 - (7-k) = 2 + k = 0 \iff k = -2$$

Then the equation is $x^2 - 6x + 9 = (x-3)^2 = 0$, so $x = 3$ and $y = 2 \cdot 3 - 2 = 4$. Check on the parabola: $9 - 12 + 7 = 4$.

Answer: $k = -2$, point of tangency $(3, 4)$.

Exercise 3 (Circles). **Level: difficult**

Find the circle with center on the x-axis passing through $A = (1, 3)$ and $B = (5, 1)$. Then write the tangent line to the circle at A.

The center is $C = (c, 0)$, with $CA = CB$:

$$(c-1)^2 + 9 = (c-5)^2 + 1 \iff c^2 - 2c + 10 = c^2 - 10c + 26 \iff 8c = 16 \iff c = 2$$

The radius is $CA = \sqrt{1+9} = \sqrt{10}$, and the circle is $(x-2)^2 + y^2 = 10$, that is $x^2 + y^2 - 4x - 6 = 0$.

The slope of CA is $\frac{3-0}{1-2} = -3$, so the tangent at A has slope $\frac{1}{3}$:

$$y - 3 = \frac{1}{3}(x-1) \iff x - 3y + 8 = 0$$

Check: the distance of C from the line is $\frac{|2+8|}{\sqrt{10}} = \sqrt{10}$, the radius.

Exercise 4 (Exponential equations). **Level: difficult**

Solve $4^x + 6^x = 2 \cdot 9^x$.

Dividing by $9^x > 0$, and since $\frac{4^x}{9^x} = \left(\frac{2}{3}\right)^{2x}$ and $\frac{6^x}{9^x} = \left(\frac{2}{3}\right)^x$, with $u = \left(\frac{2}{3}\right)^x > 0$:

$$u^2 + u - 2 = 0 \iff (u-1)(u+2) = 0 \iff u = 1 \text{ or } u = -2$$

The value $u = -2$ is not acceptable. $\left(\frac{2}{3}\right)^x = 1 \iff x = 0$.

Answer: $x = 0$.

Exercise 5 (Logarithmic equations). **Level: medium**

Solve $\log_2 x + \log_4 x + \log_8 x = 11$.

Domain: $x > 0$. Changing base, $\log_4 x = \frac{\log_2 x}{\log_2 4} = \frac{1}{2}\log_2 x$ and $\log_8 x = \frac{1}{3}\log_2 x$, so

$$\left(1 + \frac{1}{2} + \frac{1}{3}\right)\log_2 x = 11 \iff \frac{11}{6}\log_2 x = 11 \iff \log_2 x = 6 \iff x = 2^6 = 64$$

Answer: $x = 64$.

Exercise 6 (Trigonometry). **Level: difficult**

Knowing that $\tan\frac{\alpha}{2} = \frac{1}{2}$, *compute* $\sin\alpha$, $\cos\alpha$, $\tan\alpha$ *and* $\sin 3\alpha$.

With the parametric formulas, $t = \tan\frac{\alpha}{2} = \frac{1}{2}$:

$$\sin\alpha = \frac{2t}{1+t^2} = \frac{1}{5/4} = \frac{4}{5}, \qquad \cos\alpha = \frac{1-t^2}{1+t^2} = \frac{3/4}{5/4} = \frac{3}{5}, \qquad \tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{4}{3}$$

For $\sin 3\alpha$ we use the addition and duplication formulas:

$$\begin{aligned}\sin 3\alpha = \sin(2\alpha + \alpha) &= \sin 2\alpha\cos\alpha + \cos 2\alpha\sin\alpha \\ &= 2\sin\alpha\cos^2\alpha + (1 - 2\sin^2\alpha)\sin\alpha = 3\sin\alpha - 4\sin^3\alpha\end{aligned}$$

(using $\cos^2\alpha = 1 - \sin^2\alpha$). Hence $\sin 3\alpha = \frac{12}{5} - 4 \cdot \frac{64}{125} = \frac{300-256}{125} = \frac{44}{125}$.

Exercise 7 (Trigonometric equations). **Level: medium**

Solve $\cos 2x = \cos x$.

By the rule for $\cos A = \cos B$:

- $2x = x + 2k\pi \iff x = 2k\pi$;
- $2x = -x + 2k\pi \iff x = \frac{2k\pi}{3}$.

The first family is contained in the second (take k multiple of 3). In $[0, 2\pi)$ the solutions are $0, \frac{2\pi}{3}, \frac{4\pi}{3}$.

Answer: $x = \frac{2k\pi}{3}$.

Exercise 8 (Trigonometric equations)**.** **Level: difficult**

Solve $\sin x + \sqrt{3}\cos x = -\sqrt{3}$ *using the tangent of the half angle.*

Here $a = 1$, $b = \sqrt{3}$, $c = -\sqrt{3}$.

1. For $x = \pi$: $0 + \sqrt{3} \cdot (-1) = -\sqrt{3}$. So $x = \pi + 2k\pi$ are solutions (indeed $b + c = 0$).
2. For $x \neq \pi + 2k\pi$, with $t = \tan\frac{x}{2}$:

$$\frac{2t}{1+t^2} + \sqrt{3}\,\frac{1-t^2}{1+t^2} = -\sqrt{3} \iff 2t + \sqrt{3} - \sqrt{3}\,t^2 = -\sqrt{3} - \sqrt{3}\,t^2 \iff t = -\sqrt{3}$$

3. $\tan\frac{x}{2} = -\sqrt{3} \iff \frac{x}{2} = -\frac{\pi}{3} + k\pi \iff x = -\frac{2\pi}{3} + 2k\pi$.

Check for $x = -\frac{2\pi}{3}$: $-\frac{\sqrt{3}}{2} + \sqrt{3} \cdot \left(-\frac{1}{2}\right) = -\sqrt{3}$.

Answer: $x = -\frac{2\pi}{3} + 2k\pi$ or $x = \pi + 2k\pi$.

Exercise 9 (Trigonometric equations)**.** **Level: difficult**

Solve $\cos^4 x - \sin^4 x = \frac{1}{2}$.

The left side is a difference of squares:

$$\cos^4 x - \sin^4 x = (\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x) = \cos 2x \cdot 1 = \cos 2x$$

So the equation is $\cos 2x = \frac{1}{2} = \cos\frac{\pi}{3}$, that is $2x = \pm\frac{\pi}{3} + 2k\pi$.

Answer: $x = \pm\frac{\pi}{6} + k\pi$.