Classwork 13.C

Solutions

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Each exercise is marked with its level. In the solutions k denotes an arbitrary integer, $k \in \mathbb{Z}$.

Exercise 1 (Polynomial gcd)**.** **Level: medium**

Compute, with the Euclidean algorithm, the gcd of

$$A(x) = x^4 + x^2 + 1 \qquad \textit{and} \qquad B(x) = x^3 - 1$$

First division, A by B:

$$\begin{array}{rr|l} x^4 + x^2 & +1 & x^3 - 1 \\ \cline{3-3} -x^4 & +x & x \\ \cline{1-2} & x^2 + x + 1 & \end{array}$$

The remainder is $R_1(x) = x^2 + x + 1$. Second division, B by R_1:

$$\begin{array}{rr|l} x^3 & -1 & x^2 + x + 1 \\ \cline{3-3} -x^3 - x^2 - x & & x - 1 \\ \cline{1-1} -x^2 - x - 1 & & \\ x^2 + x + 1 & & \\ \cline{1-1} 0 & & \end{array}$$

The remainder is 0, so the gcd is $x^2 + x + 1$.

Check by factoring: $B(x) = (x-1)(x^2+x+1)$ and $A(x) = (x^2+1)^2 - x^2 = (x^2+x+1)(x^2-x+1)$.

Answer: $\gcd(A, B) = x^2 + x + 1$.

Exercise 2 (Polynomial equations)**.** **Level: medium**

Solve $x^3 - 3x - 2 = 0$.

The possible rational roots are the divisors of 2: $\pm 1, \pm 2$. Since $P(-1) = 0$, the polynomial $P(x)$ is divisible by $x + 1$ (factor theorem). Dividing:

$$\begin{array}{rr|l} x^3 & -3x - 2 & x + 1 \\ \cline{3-3} -x^3 - x^2 & & x^2 - x - 2 \\ \cline{1-1} -x^2 - 3x & & \\ x^2 + x & & \\ \cline{1-1} -2x - 2 & & \\ 2x + 2 & & \\ \cline{1-1} 0 & & \end{array}$$

so $P(x) = (x+1)(x^2-x-2)$. Then $x^2-x-2 = 0 \iff (x-2)(x+1) = 0$, so $P(x) = (x+1)^2(x-2)$: $x = -1$ is a double root.

Answer: $x = -1$ (double), $x = 2$.

Exercise 3 (Parabolas). **Level: easy**

For the parabola $y = 2x^2 - 8x + 6$ find the vertex, the points where it meets the coordinate axes, and draw it.

Here $a = 2 > 0$: the parabola opens upwards.

- Vertex: $x_V = -\frac{b}{2a} = 2$, $y_V = 8 - 16 + 6 = -2$, so $V = (2, -2)$.
- y-axis: for $x = 0$, $y = 6$, so the point is $(0, 6)$.
- x-axis: $2x^2 - 8x + 6 = 0 \iff x^2 - 4x + 3 = 0 \iff (x-1)(x-3) = 0$, so the points are $(1, 0)$ and $(3, 0)$.

With the vertex and these points we can draw the graph.

Exercise 4 (Circles). **Level: medium**

Write the equation of the circle having the segment AB as a diameter, with $A = (-1, 3)$ and $B = (5, -5)$. Is the origin inside or outside the circle?

The center is the midpoint of AB, $C = \left(\frac{-1+5}{2}, \frac{3-5}{2}\right) = (2, -1)$, and the radius is half the length of AB:

$$AB = \sqrt{6^2 + 8^2} = 10, \qquad r = 5$$

The circle is $(x-2)^2 + (y+1)^2 = 25$, that is $x^2 + y^2 - 4x + 2y - 20 = 0$.

The distance of the origin from the center is $\sqrt{4+1} = \sqrt{5} < 5$: the origin is inside the circle (equivalently, substituting $(0, 0)$ in the left side we get $-20 < 0$).

Exercise 5 (Exponential equations). **Level: medium**

Solve $2^{x+2} + 2^{x-1} = 36$.

Since $2^{x+2} = 4 \cdot 2^x$ and $2^{x-1} = \frac{1}{2} \cdot 2^x$, we can collect 2^x:

$$2^x \left(4 + \frac{1}{2}\right) = 36 \iff \frac{9}{2} \cdot 2^x = 36 \iff 2^x = 8 = 2^3 \iff x = 3$$

Check: $2^5 + 2^2 = 32 + 4 = 36$.

Answer: $x = 3$.

Exercise 6 (Logarithmic equations). **Level: medium**

Solve $(\log_2 x)^2 - 3\log_2 x + 2 = 0$.

Domain: $x > 0$. With $t = \log_2 x$ (any real value):

$$t^2 - 3t + 2 = 0 \iff (t-1)(t-2) = 0 \iff t = 1 \text{ or } t = 2$$

$\log_2 x = 1 \iff x = 2$ and $\log_2 x = 2 \iff x = 4$, both in the domain.

Answer: $x = 2$ or $x = 4$.

Exercise 7 (Trigonometry). **Level: easy**

Using the subtraction formulas and $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$, compute $\cos\frac{\pi}{12}$ and $\sin\frac{\pi}{12}$.

$$\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4} = \frac{1}{2}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}+\sqrt{6}}{4}$$

$$\sin\frac{\pi}{12} = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4} = \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

Check: $\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)^2 + \left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)^2 = \frac{(8+2\sqrt{12})+(8-2\sqrt{12})}{16} = 1$.

Exercise 8 (Trigonometric equations). **Level: easy**

Solve $2\sin 2x = 1$.

We have $\sin 2x = \frac{1}{2} = \sin\frac{\pi}{6}$, so

$$2x = \frac{\pi}{6} + 2k\pi \quad \text{or} \quad 2x = \frac{5\pi}{6} + 2k\pi \iff x = \frac{\pi}{12} + k\pi \quad \text{or} \quad x = \frac{5\pi}{12} + k\pi$$

Answer: $x = \frac{\pi}{12} + k\pi$ or $x = \frac{5\pi}{12} + k\pi$.

Exercise 9 (Trigonometric equations). **Level: medium**

Solve $\cos x - \sqrt{3}\sin x = -1$ *using the tangent of the half angle.*

Writing it as $-\sqrt{3}\sin x + \cos x = -1$: $a = -\sqrt{3}$, $b = 1$, $c = -1$.

1. For $x = \pi$: $-1 - 0 = -1$. So $x = \pi + 2k\pi$ are solutions (indeed $b + c = 0$).
2. For $x \neq \pi + 2k\pi$, with $t = \tan\frac{x}{2}$:

$$\frac{1-t^2}{1+t^2} - \sqrt{3}\,\frac{2t}{1+t^2} = -1 \iff 1 - t^2 - 2\sqrt{3}\,t = -1 - t^2 \iff 2\sqrt{3}\,t = 2 \iff t = \frac{\sqrt{3}}{3}$$

   (the term in t^2 cancels, since $x = \pi$ is a solution).
3. $\tan\frac{x}{2} = \frac{\sqrt{3}}{3} \iff \frac{x}{2} = \frac{\pi}{6} + k\pi \iff x = \frac{\pi}{3} + 2k\pi$.

Check for $x = \frac{\pi}{3}$: $\frac{1}{2} - \sqrt{3}\cdot\frac{\sqrt{3}}{2} = \frac{1}{2} - \frac{3}{2} = -1$.

Answer: $x = \frac{\pi}{3} + 2k\pi$ or $x = \pi + 2k\pi$.

Exercise 10 (Exponential equations). **Level: VERY DIFFICULT**

Solve $\left(\sqrt{2+\sqrt{3}}\right)^x + \left(\sqrt{2-\sqrt{3}}\right)^x = 4$.

The key remark: $(2+\sqrt{3})(2-\sqrt{3}) = 4 - 3 = 1$, so $\sqrt{2-\sqrt{3}} = \dfrac{1}{\sqrt{2+\sqrt{3}}}$, and

$$\left(\sqrt{2-\sqrt{3}}\right)^x = \frac{1}{\left(\sqrt{2+\sqrt{3}}\right)^x}$$

Set $u = \left(\sqrt{2+\sqrt{3}}\right)^x > 0$. Multiplying by u:

$$u + \frac{1}{u} = 4 \iff u^2 - 4u + 1 = 0 \iff u = 2 \pm \sqrt{3}$$

- $u = 2 + \sqrt{3} = \left(\sqrt{2+\sqrt{3}}\right)^2 \iff x = 2$;
- $u = 2 - \sqrt{3} = \dfrac{1}{2+\sqrt{3}} = \left(\sqrt{2+\sqrt{3}}\right)^{-2} \iff x = -2$.

(The function $x \mapsto \left(\sqrt{2+\sqrt{3}}\right)^x$ is injective, since the base is greater than 1.) Check for $x = 2$: $(2+\sqrt{3}) + (2-\sqrt{3}) = 4$.

Answer: $x = 2$ or $x = -2$.