Classwork 13.B

Solutions

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Each exercise is marked with its level. In the solutions k denotes an arbitrary integer, $k \in \mathbb{Z}$.

Exercise 1 (Polynomial gcd). **Level: medium**

Compute, with the Euclidean algorithm, the gcd of

$$A(x) = x^4 - 5x^2 + 4 \qquad \text{and} \qquad B(x) = x^3 - 3x^2 + 2x$$

First division, A by B:

$$\begin{array}{rr|l} x^4 - 5x^2 & +4 & x^3 - 3x^2 + 2x \\ \cline{3-3} -x^4 + 3x^3 - 2x^2 & & x + 3 \\ \cline{1-1} 3x^3 - 7x^2 & & \\ -3x^3 + 9x^2 - 6x & & \\ \cline{1-1} 2x^2 - 6x + 4 & & \end{array}$$

The remainder is $R_1(x) = 2x^2 - 6x + 4$. Second division, B by R_1:

$$\begin{array}{r|l} x^3 - 3x^2 + 2x & 2x^2 - 6x + 4 \\ \cline{2-2} -x^3 + 3x^2 - 2x & \frac{1}{2}x \\ \cline{1-1} 0 & \end{array}$$

The remainder is 0, so the gcd is $2x^2 - 6x + 4$, or, choosing the monic one, $x^2 - 3x + 2$.

Check by factoring: $A(x) = (x^2 - 1)(x^2 - 4) = (x - 1)(x + 1)(x - 2)(x + 2)$ and $B(x) = x(x - 1)(x - 2)$.

Answer: $\gcd(A, B) = x^2 - 3x + 2$.

Exercise 2 (Polynomial equations). **Level: medium**

Solve $x^4 - 10x^2 + 9 = 0$.

This is a biquadratic equation: with $t = x^2$,

$$t^2 - 10t + 9 = 0 \iff (t - 1)(t - 9) = 0 \iff t = 1 \text{ or } t = 9$$

Then $x^2 = 1 \iff x = \pm 1$ and $x^2 = 9 \iff x = \pm 3$. Equivalently, $x^4 - 10x^2 + 9 = (x - 1)(x + 1)(x - 3)(x + 3)$.

Answer: $x = \pm 1$, $x = \pm 3$.

Exercise 3 (Parabolas). **Level: easy**

Find the parabola $y = ax^2 + bx + c$ *passing through the points* $(0, 1)$, $(1, 0)$ *and* $(2, 1)$, *and find its vertex.*

Imposing the three points:

$$\begin{cases} c = 1 \\ a + b + c = 0 \\ 4a + 2b + c = 1 \end{cases} \iff \begin{cases} c = 1 \\ a + b = -1 \\ 4a + 2b = 0 \end{cases} \iff \begin{cases} c = 1 \\ b = -2a \\ a - 2a = -1 \end{cases} \iff \begin{cases} a = 1 \\ b = -2 \\ c = 1 \end{cases}$$

The parabola is $y = x^2 - 2x + 1 = (x - 1)^2$, and its vertex is $V = (1, 0)$ (indeed $(x - 1)^2 \geq 0$, with equality only for $x = 1$).

Exercise 4 (Circles). **Level: medium**

Find the center and the radius of the circle $x^2+y^2+2x-4y-20=0$*, and its intersections with the line* $r\colon 4x-3y+10=0$.

Completing the squares,

$$(x+1)^2+(y-2)^2=20+1+4=25$$

so the center is $C=(-1,2)$ and the radius is 5.

From r, $y=\frac{4x+10}{3}$. Substituting and multiplying by 9:

$$9x^2+(4x+10)^2+18x-12(4x+10)-180=0 \iff 25x^2+50x-200=0 \iff x^2+2x-8=0$$

so $x=2$ or $x=-4$, and the intersections are $(2,6)$ and $(-4,-2)$. Notice that r passes through the center $(4\cdot(-1)-3\cdot 2+10=0)$: the segment joining the two points is a diameter, and indeed its length is $\sqrt{36+64}=10$.

Exercise 5 (Exponential equations). **Level: medium**

Solve $3^{x+1}+3^{1-x}=10$.

Since $3^{x+1}=3\cdot 3^x$ and $3^{1-x}=\frac{3}{3^x}$, with $t=3^x>0$:

$$3t+\frac{3}{t}=10 \iff 3t^2-10t+3=0 \iff t=\frac{10\pm 8}{6} \iff t=3 \text{ or } t=\frac{1}{3}$$

(multiplying by $t>0$). $3^x=3 \iff x=1$ and $3^x=\frac{1}{3} \iff x=-1$.

Answer: $x=1$ or $x=-1$.

Exercise 6 (Logarithmic equations). **Level: medium**

Solve $\log_2 x-\log_2(x-3)=2$.

Domain: $x>0$ and $x-3>0$, that is $x>3$. In the domain, using $\log_2 a-\log_2 b=\log_2\frac{a}{b}$,

$$\log_2\frac{x}{x-3}=2 \iff \frac{x}{x-3}=4 \iff x=4x-12 \iff x=4$$

and $4>3$ is in the domain.

Answer: $x=4$.

Exercise 7 (Trigonometry). **Level: easy**

Knowing that $\sin\alpha=-\frac{4}{5}$ *and* $\frac{3\pi}{2}<\alpha<2\pi$*, compute* $\cos\alpha$*,* $\tan\alpha$*,* $\sin 2\alpha$ *and* $\cos 2\alpha$.

From $\cos^2\alpha=1-\frac{16}{25}=\frac{9}{25}$, and $\cos\alpha>0$ in the fourth quadrant:

$$\cos\alpha=\frac{3}{5}, \qquad \tan\alpha=\frac{\sin\alpha}{\cos\alpha}=-\frac{4}{3}$$

With the duplication formulas

$$\sin 2\alpha=2\cdot\left(-\frac{4}{5}\right)\cdot\frac{3}{5}=-\frac{24}{25}, \qquad \cos 2\alpha=\cos^2\alpha-\sin^2\alpha=\frac{9-16}{25}=-\frac{7}{25}$$

Exercise 8 (Trigonometric equations). **Level: easy**

Solve $\tan^2 x=3$.

Domain: $x\neq\frac{\pi}{2}+k\pi$. We have $\tan x=\sqrt{3}$ or $\tan x=-\sqrt{3}$.

- $\tan x=\sqrt{3}=\tan\frac{\pi}{3} \iff x=\frac{\pi}{3}+k\pi$;
- $\tan x=-\sqrt{3}=\tan\left(-\frac{\pi}{3}\right) \iff x=-\frac{\pi}{3}+k\pi$.

Answer: $x=\pm\frac{\pi}{3}+k\pi$.

Exercise 9 (Trigonometric equations). **Level: medium**

Solve $\sqrt{3}\sin x - \cos x = -1$ *using the tangent of the half angle.*

Here $a = \sqrt{3}$, $b = -1$, $c = -1$.

1. For $x = \pi$: $0 - (-1) = 1 \neq -1$. The values $x = \pi + 2k\pi$ are not solutions.
2. For $x \neq \pi + 2k\pi$, with $t = \tan\frac{x}{2}$:

$$\sqrt{3}\,\frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2} = -1 \iff 2\sqrt{3}\,t - 1 + t^2 = -1 - t^2$$
$$\iff 2t^2 + 2\sqrt{3}\,t = 0 \iff 2t\left(t + \sqrt{3}\right) = 0$$

   so $t = 0$ or $t = -\sqrt{3}$.

3. $\tan\frac{x}{2} = 0 \iff x = 2k\pi$; $\quad \tan\frac{x}{2} = -\sqrt{3} \iff \frac{x}{2} = -\frac{\pi}{3} + k\pi \iff x = -\frac{2\pi}{3} + 2k\pi$.

Check for $x = -\frac{2\pi}{3}$: $\sqrt{3}\cdot\left(-\frac{\sqrt{3}}{2}\right) - \left(-\frac{1}{2}\right) = -\frac{3}{2} + \frac{1}{2} = -1$.

Answer: $x = 2k\pi$ or $x = -\frac{2\pi}{3} + 2k\pi$.

Exercise 10 (Exponentials and trigonometry). **Level: VERY DIFFICULT**

Solve $9^{\sin x} + 9^{-\sin x} = \frac{10}{3}$.

Set $u = 9^{\sin x} > 0$; then $9^{-\sin x} = \frac{1}{u}$ and, multiplying by $3u > 0$,

$$u + \frac{1}{u} = \frac{10}{3} \iff 3u^2 - 10u + 3 = 0 \iff u = \frac{10 \pm 8}{6} \iff u = 3 \text{ or } u = \frac{1}{3}$$

- $9^{\sin x} = 3 = 9^{1/2} \iff \sin x = \frac{1}{2} \iff x = \frac{\pi}{6} + 2k\pi$ or $x = \frac{5\pi}{6} + 2k\pi$;
- $9^{\sin x} = \frac{1}{3} = 9^{-1/2} \iff \sin x = -\frac{1}{2} \iff x = -\frac{\pi}{6} + 2k\pi$ or $x = \frac{7\pi}{6} + 2k\pi$.

Together, $x = \frac{\pi}{6} + k\pi$ or $x = \frac{5\pi}{6} + k\pi$, that is $x = \pm\frac{\pi}{6} + k\pi$.

Answer: $x = \pm\frac{\pi}{6} + k\pi$.