# Classwork 13.A

Solutions

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Each exercise is marked with its level. In the solutions $k$ denotes an arbitrary integer, $k \in \mathbb{Z}$.

**Exercise 1** (Polynomial gcd)**.** **Level: medium**

*Compute, with the Euclidean algorithm, the gcd of*

$$A(x) = x^3 + x^2 - x - 1 \qquad \textit{and} \qquad B(x) = x^3 - x^2 - x + 1$$

First division, $A$ by $B$:

$$\begin{array}{r|l} x^3 + x^2 - x - 1 & x^3 - x^2 - x + 1 \\ \hline -x^3 + x^2 + x - 1 & 1 \\ \hline 2x^2 \quad -2 & \end{array}$$

The remainder is $R_1(x) = 2x^2 - 2$. Second division, $B$ by $R_1$:

$$\begin{array}{r|l} x^3 - x^2 - x + 1 & 2x^2 - 2 \\ \hline -x^3 \quad + x & \frac{1}{2}x - \frac{1}{2} \\ \hline -x^2 \quad + 1 & \\ x^2 \quad - 1 & \\ \hline 0 & \end{array}$$

The remainder is 0, so the gcd is $2x^2 - 2$, or, choosing the monic one, $x^2 - 1$.

Check by factoring: $A(x) = x^2(x+1) - (x+1) = (x-1)(x+1)^2$ and $B(x) = x^2(x-1) - (x-1) = (x-1)^2(x+1)$.

**Answer:** $\gcd(A, B) = x^2 - 1$.

**Exercise 2** (Polynomial equations)**.** **Level: medium**

*Solve* $2x^3 - 3x^2 - 3x + 2 = 0$.

The possible rational roots are $\pm 1, \pm 2, \pm\frac{1}{2}$ (divisors of 2 over divisors of 2). Since $P(2) = 0$, the polynomial $P(x)$ is divisible by $x - 2$ (factor theorem). Dividing:

$$\begin{array}{r|l} 2x^3 - 3x^2 - 3x + 2 & x - 2 \\ \hline -2x^3 + 4x^2 & 2x^2 + x - 1 \\ \hline x^2 - 3x & \\ -x^2 + 2x & \\ \hline -x + 2 & \\ x - 2 & \\ \hline 0 & \end{array}$$

so $P(x) = (x-2)(2x^2 + x - 1)$. Then $2x^2 + x - 1 = 0 \iff x = \frac{-1 \pm 3}{4}$, that is $x = \frac{1}{2}$ or $x = -1$.

**Answer:** $x = 2$, $x = \frac{1}{2}$, $x = -1$.

**Exercise 3** (Parabolas). **Level: easy**

*For the parabola $y = x^2 + 2x - 8$ find the vertex, the points where it meets the coordinate axes, and draw it.*

Here $a = 1 > 0$: the parabola opens upwards.

- Vertex: $x_V = -\frac{b}{2a} = -1$, $y_V = 1 - 2 - 8 = -9$, so $V = (-1, -9)$.
- $y$-axis: for $x = 0$, $y = -8$, so the point is $(0, -8)$.
- $x$-axis: $x^2 + 2x - 8 = 0 \iff (x + 4)(x - 2) = 0$, so the points are $(-4, 0)$ and $(2, 0)$.

With the vertex and these points we can draw the graph.

**Exercise 4** (Circles). **Level: medium**

*Write the equation of the circle passing through the points $O = (0, 0)$, $A = (6, 0)$ and $B = (0, 8)$, and find its center and radius.*

Write the circle as $x^2 + y^2 + ax + by + c = 0$ and impose the three points:

$$\begin{cases} c = 0 & (O) \\ 36 + 6a + c = 0 & (A) \\ 64 + 8b + c = 0 & (B) \end{cases} \iff \begin{cases} c = 0 \\ a = -6 \\ b = -8 \end{cases}$$

The circle is $x^2 + y^2 - 6x - 8y = 0$. Completing the squares, $(x - 3)^2 + (y - 4)^2 = 25$: the center is $C = (3, 4)$ and the radius is 5.

Remark: the angle $\widehat{AOB}$ is a right angle, so $AB$ is a diameter: its midpoint is $(3, 4)$ and its length is $\sqrt{36 + 64} = 10$.

**Exercise 5** (Exponential equations). **Level: medium**

*Solve* $2^{2x+1} - 5 \cdot 2^x + 2 = 0$.

Since $2^{2x+1} = 2 \cdot (2^x)^2$, with $t = 2^x > 0$ the equation becomes

$$2t^2 - 5t + 2 = 0 \iff t = \frac{5 \pm 3}{4} \iff t = 2 \text{ or } t = \frac{1}{2}$$

Both values are positive. $2^x = 2 \iff x = 1$ and $2^x = \frac{1}{2} = 2^{-1} \iff x = -1$.

**Answer:** $x = 1$ or $x = -1$.

**Exercise 6** (Logarithmic equations). **Level: medium**

*Solve* $\log_3(x + 2) + \log_3 x = 1$.

Domain: $x + 2 > 0$ and $x > 0$, that is $x > 0$. In the domain

$$\log_3\big(x(x + 2)\big) = 1 \iff x^2 + 2x = 3 \iff x^2 + 2x - 3 = 0 \iff (x + 3)(x - 1) = 0$$

The value $x = -3$ is not in the domain.

**Answer:** $x = 1$.

**Exercise 7** (Trigonometry). **Level: easy**

*Knowing that* $\tan\alpha = \frac{3}{4}$ *and* $\pi < \alpha < \frac{3\pi}{2}$, *compute* $\sin\alpha$, $\cos\alpha$, $\sin 2\alpha$ *and* $\cos 2\alpha$.

From $1 + \tan^2\alpha = \frac{1}{\cos^2\alpha}$ we get $\cos^2\alpha = \frac{1}{1 + \frac{9}{16}} = \frac{16}{25}$. In the third quadrant $\cos\alpha < 0$ and $\sin\alpha < 0$, so

$$\cos\alpha = -\frac{4}{5}, \qquad \sin\alpha = \tan\alpha\cos\alpha = \frac{3}{4} \cdot \left(-\frac{4}{5}\right) = -\frac{3}{5}$$

With the duplication formulas

$$\sin 2\alpha = 2\sin\alpha\cos\alpha = 2 \cdot \left(-\frac{3}{5}\right)\left(-\frac{4}{5}\right) = \frac{24}{25}, \qquad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \frac{16 - 9}{25} = \frac{7}{25}$$

**Exercise 8** (Trigonometric equations). **Level: easy**

*Solve* $2\cos^2 x - \cos x = 0$.

Factoring, $\cos x\,(2\cos x - 1) = 0$.

- $\cos x = 0 \iff x = \frac{\pi}{2} + k\pi$;
- $\cos x = \frac{1}{2} = \cos\frac{\pi}{3} \iff x = \pm\frac{\pi}{3} + 2k\pi$.

**Answer:** $x = \frac{\pi}{2} + k\pi$ or $x = \pm\frac{\pi}{3} + 2k\pi$.

**Exercise 9** (Trigonometric equations). **Level: medium**

*Solve* $\sin x + \cos x = 1$ *using the tangent of the half angle.*

Here $a = 1$, $b = 1$, $c = 1$.

1. For $x = \pi$: $0 + (-1) = -1 \neq 1$. The values $x = \pi + 2k\pi$ are not solutions.
2. For $x \neq \pi + 2k\pi$, with $t = \tan\frac{x}{2}$:

$$\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} = 1 \iff 2t + 1 - t^2 = 1 + t^2 \iff 2t^2 - 2t = 0 \iff 2t(t-1) = 0$$

   so $t = 0$ or $t = 1$.

3. $\tan\frac{x}{2} = 0 \iff x = 2k\pi$; $\quad \tan\frac{x}{2} = 1 \iff \frac{x}{2} = \frac{\pi}{4} + k\pi \iff x = \frac{\pi}{2} + 2k\pi$.

**Answer:** $x = 2k\pi$ or $x = \frac{\pi}{2} + 2k\pi$.

**Exercise 10** (Logarithms and trigonometry). **Level: VERY DIFFICULT**

*Solve* $\log_2(1 + \cos x) + \log_2(1 - \cos x) = -2$.

*Domain.* We need $1 + \cos x > 0$ and $1 - \cos x > 0$, that is $\cos x \neq -1$ and $\cos x \neq 1$, that is $\sin x \neq 0$: $x \neq k\pi$.

In the domain, using $\log_2 a + \log_2 b = \log_2(ab)$ and $(1 + \cos x)(1 - \cos x) = 1 - \cos^2 x = \sin^2 x$,

$$\log_2\left(\sin^2 x\right) = -2 \iff \sin^2 x = 2^{-2} = \frac{1}{4} \iff \sin x = \pm\frac{1}{2}$$

From $\sin x = \frac{1}{2}$: $x = \frac{\pi}{6} + 2k\pi$ or $x = \frac{5\pi}{6} + 2k\pi$; from $\sin x = -\frac{1}{2}$: $x = -\frac{\pi}{6} + 2k\pi$ or $x = \frac{7\pi}{6} + 2k\pi$. Together, $x = \frac{\pi}{6} + k\pi$ or $x = \frac{5\pi}{6} + k\pi$. All these values satisfy $\sin x \neq 0$, so they are in the domain.

**Answer:** $x = \frac{\pi}{6} + k\pi$ or $x = \frac{5\pi}{6} + k\pi$.