Math 0 Test

FCS UNIPI - Math

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1. Is it true that $\sqrt{5}+1+\sqrt{2}>2+\sqrt{3}$?

   Solution. Numerically, $\sqrt{5}\approx 2.236$, $\sqrt{2}\approx 1.414$, $\sqrt{3}\approx 1.732$.

   $$\text{LHS}\approx 2.236+1+1.414=4.650,\qquad \text{RHS}\approx 2+1.732=3.732.$$

   Since $4.650>3.732$, the statement is **True**.

2. Solve, considering existence conditions, the equation $\dfrac{\sqrt{x^2+4}}{x^2}=0$.

   Solution. The domain requires $x^2\neq 0$, i.e. $x\neq 0$. For every real x, $\sqrt{x^2+4}\geq\sqrt{4}=2>0$, so the numerator is never 0. Hence the fraction is never 0. **Answer:** $\emptyset$ (no solution).

3. Simplify $\dfrac{p^2-q^2}{p-q}+3$.

   Solution. Since $p\neq q$, factor the numerator: $p^2-q^2=(p-q)(p+q)$, so

   $$\frac{(p-q)(p+q)}{p-q}+3=(p+q)+3=p+q+3.$$

   Answer: $p+q+3$.

4. Find the greatest common divisor and the least common multiple of the integers 84, 126.

   Solution. Factor into primes: $84=2^2\cdot 3\cdot 7$, $126=2\cdot 3^2\cdot 7$.

   $$\gcd=2\cdot 3\cdot 7=42,\qquad \operatorname{lcm}=2^2\cdot 3^2\cdot 7=252.$$

   Answer: $\gcd=42$, $\operatorname{lcm}=252$.

5. Solve $35x-21=0$.

   Solution. $35x=21\Rightarrow x=\dfrac{21}{35}=\dfrac{3}{5}$. **Answer:** $x=\dfrac{3}{5}$.

6. Solve $x^2+2x-15=0$.

   Solution. Factor: we need two numbers with sum 2 and product -15: these are 5 and -3.

   $$x^2+2x-15=(x+5)(x-3)=0\ \Rightarrow\ x=-5 \text{ or } x=3.$$

   Answer: $x=-5, 3$.

7. Solve $7-2x>x$.

   Solution. $7>3x\Rightarrow x<\dfrac{7}{3}$. **Answer:** $x<\dfrac{7}{3}$.

8. Solve $x^2 + 2x - 15 < 0$.

   Solution. From Exercise 8, $x^2+2x-15 = (x+5)(x-3)$, which is negative between its roots. **Answer:** $-5 < x < 3$.

9. Solve, considering existence conditions, $\dfrac{x+3}{x-2} \le 0$.

   Solution. Domain: $x \neq 2$. The numerator $x+3 = 0$ at $x = -3$; the denominator changes sign at $x = 2$. Sign study: the fraction is ≤ 0 when the numerator and denominator have opposite signs (or numerator $= 0$), i.e. for $-3 \le x < 2$. **Answer:** $x \in [-3, 2)$.

10. Simplify $\dfrac{x^2-3x-10}{x^2-2x-15}$.

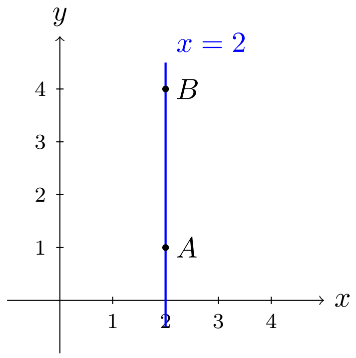
    Solution. Factor both: $x^2 - 3x - 10 = (x-5)(x+2)$ and $x^2 - 2x - 15 = (x-5)(x+3)$.

    $$\frac{(x-5)(x+2)}{(x-5)(x+3)} = \frac{x+2}{x+3} \qquad (x \neq 5).$$

    Answer: $\dfrac{x+2}{x+3}$.

11. Draw on the Cartesian plane the line through $A : (2, 1)$, $B : (2, 4)$.

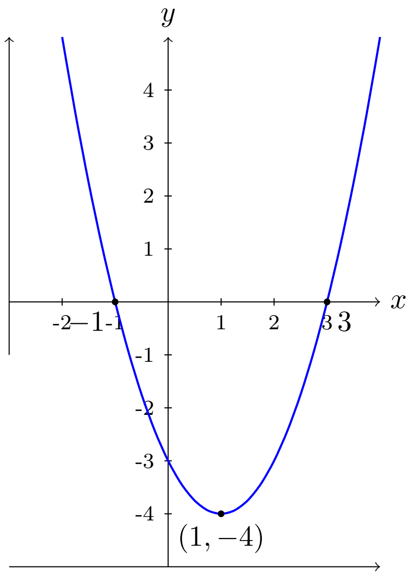
    Solution. Both points have $x = 2$, so the line is vertical: $x = 2$.

    Answer: the vertical line $x = 2$.

12. Draw on the Cartesian plane the parabola $f(x) = x^2 - 2x - 3$.

    Solution. Roots: $x^2 - 2x - 3 = (x-3)(x+1) = 0 \Rightarrow x = -1, 3$. Vertex at $x = 1$, $f(1) = 1 - 2 - 3 = -4$, so vertex $(1, -4)$.



Answer: upward-opening parabola with roots $-1, 3$ and vertex $(1, -4)$.

13. Simplify and calculate

$$\left(2 - \frac{3}{4}\right) \cdot \frac{4}{5} \cdot \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{6}\right) - \left(\frac{5}{4} - \frac{2}{3}\right).$$

Solution. Step by step:

$$2 - \frac{3}{4} = \frac{5}{4}, \qquad \frac{5}{4} \cdot \frac{4}{5} \cdot \frac{1}{2} = \frac{5 \cdot 4}{4 \cdot 5} \cdot \frac{1}{2} = 1 \cdot \frac{1}{2} = \frac{1}{2}.$$

$$\frac{1}{3} + \frac{1}{6} = \frac{2}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}.$$

$$\frac{5}{4} - \frac{2}{3} = \frac{15}{12} - \frac{8}{12} = \frac{7}{12}.$$

Putting it together:

$$\frac{1}{2} + \frac{1}{2} - \frac{7}{12} = 1 - \frac{7}{12} = \frac{5}{12}.$$

Answer: $\frac{5}{12}$.

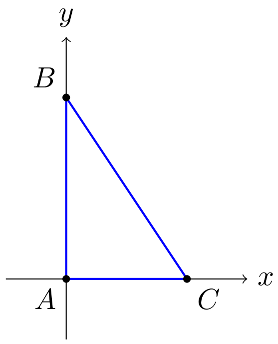
14. Draw on the Cartesian plane the triangle ABC, where $A = (0, 0)$, $B = (0, 3)$, $C = (2, 0)$. Find the area and the perimeter of the triangle.

Solution. This is a right triangle with legs $AB = 3$ and $AC = 2$ along the axes.

$$\text{Area} = \frac{1}{2} \cdot 3 \cdot 2 = 3.$$

Hypotenuse: $BC = \sqrt{(2-0)^2 + (0-3)^2} = \sqrt{4+9} = \sqrt{13}$.

$$\text{Perimeter} = 3 + 2 + \sqrt{13} = 5 + \sqrt{13}.$$



Answer: Area $= 3$, Perimeter $= 5 + \sqrt{13}$.

15. Draw on the Cartesian plane the triangle ABC, where $A = (-2, 1)$, $B = (0, 4)$, $C = (3, 1)$. Find the area and the perimeter of the triangle.

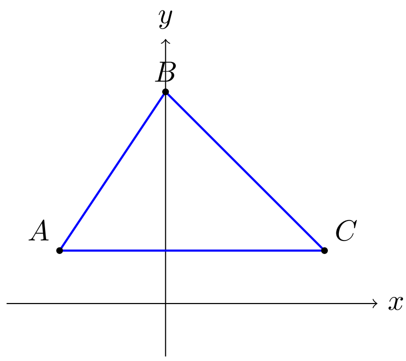
Solution. Area (shoelace formula):

$$\text{Area} = \frac{1}{2}\left|x_A(y_B - y_C) + x_B(y_C - y_A) + x_C(y_A - y_B)\right| = \frac{1}{2}\left|-2(4-1) + 0(1-1) + 3(1-4)\right|$$

$$= \frac{1}{2}\left|-6 + 0 - 9\right| = \frac{1}{2} \cdot 15 = \frac{15}{2}.$$

Side lengths:

$$AB = \sqrt{(0-(-2))^2 + (4-1)^2} = \sqrt{4+9} = \sqrt{13},$$
$$BC = \sqrt{(3-0)^2 + (1-4)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2},$$
$$AC = \sqrt{(3-(-2))^2 + (1-1)^2} = \sqrt{25} = 5.$$
$$\text{Perimeter} = 5 + \sqrt{13} + 3\sqrt{2}.$$

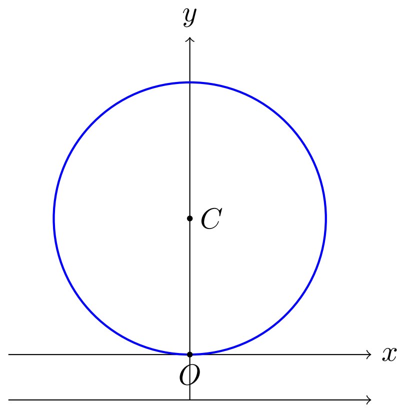


Answer: Area $= \frac{15}{2}$, Perimeter $= 5 + \sqrt{13} + 3\sqrt{2}$.

16. Compute the area and perimeter (circumference) of the circle with center $C : (0, 3)$ that intersects the line $x = 0$ at the origin $O : (0, 0)$.

Solution. The radius is the distance from C to O: $r = \sqrt{(0-0)^2 + (3-0)^2} = 3$.

$$\text{Area} = \pi r^2 = 9\pi, \qquad \text{Circumference} = 2\pi r = 6\pi.$$



Answer: Area $= 9\pi$, Circumference $= 6\pi$.

17. Put in rational standard form $\dfrac{\sqrt{3}}{4-\sqrt{3}}$.

    Solution. Multiply numerator and denominator by the conjugate $4+\sqrt{3}$:

    $$\frac{\sqrt{3}}{4-\sqrt{3}} \cdot \frac{4+\sqrt{3}}{4+\sqrt{3}} = \frac{4\sqrt{3}+3}{16-3} = \frac{4\sqrt{3}+3}{13} = \frac{3+4\sqrt{3}}{13}.$$

    Answer: $\dfrac{3+4\sqrt{3}}{13}$.

18. Given the function $f(x) = \sqrt{x^2-4}$, find $f(2)$ and $f^{-1}(0)$.

    Solution. $f(2) = \sqrt{4-4} = \sqrt{0} = 0$. For $f^{-1}(0)$: solve $\sqrt{x^2-4} = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$. **Answer:** $f(2) = 0$, $f^{-1}(0) = \{\pm 2\}$.

19. Solve, considering existence conditions, the equation $\dfrac{x^3-x^2-9x+9}{x-3} = 0$.

    Solution. Domain: $x \neq 3$. Factor the numerator by grouping:

    $$x^3-x^2-9x+9 = x^2(x-1) - 9(x-1) = (x-1)(x^2-9) = (x-1)(x-3)(x+3).$$

    For $x \neq 3$, the equation simplifies to $(x-1)(x+3) = 0$. **Answer:** $x = 1, -3$.

20. Find the greatest common divisor and the least common multiple of the polynomials x^2-4, x^2+x-2.

    Solution. Factor: $x^2-4 = (x-2)(x+2)$, $x^2+x-2 = (x+2)(x-1)$.

    $$\gcd = x+2.$$

    $$\text{lcm} = (x-2)(x+2)(x-1) = (x^2-4)(x-1) = x^3-x^2-4x+4.$$

    Answer: $\gcd = x+2$, $\text{lcm} = x^3-x^2-4x+4$.

21. Solve graphically the inequality $3^x + x - 1 > 0$.

    Solution. Rewrite as $3^x > 1-x$. Compare $y = 3^x$ (increasing exponential) and $y = 1-x$ (decreasing line). At $x = 0$: $3^0 = 1 = 1-0$, so the curves meet at $x = 0$. For $x > 0$, $3^x > 1$ and $1-x < 1$, so $3^x > 1-x$; for $x < 0$ the opposite holds. **Answer:** $x > 0$.

22. Compute the exact value of $\sin(30^\circ) + \cos(60^\circ)$.

Solution. Recall $\sin(30^\circ) = \frac{1}{2}$ and $\cos(60^\circ) = \frac{1}{2}$.

$$\sin(30^\circ) + \cos(60^\circ) = \frac{1}{2} + \frac{1}{2} = 1.$$

Answer: 1.

23. Solve $\sin(x) = \frac{1}{2}$ for $x \in [0, 2\pi)$.

Solution. The reference angle with $\sin = \frac{1}{2}$ is $\frac{\pi}{6}$. Sine is positive in the first and second quadrants, so

$$x = \frac{\pi}{6} \quad \text{or} \quad x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}.$$

Answer: $x = \frac{\pi}{6}, \frac{5\pi}{6}$.

24. Given $\sin(x) = \frac{3}{5}$ with x in the first quadrant ($0 < x < \pi/2$), find $\cos(x)$ and $\cos(2x)$.

Solution. From $\sin^2 x + \cos^2 x = 1$:

$$\cos^2 x = 1 - \frac{9}{25} = \frac{16}{25} \;\Rightarrow\; \cos x = \frac{4}{5} \quad \text{(positive, since } x \text{ is in the first quadrant).}$$

Using $\cos(2x) = \cos^2 x - \sin^2 x$:

$$\cos(2x) = \frac{16}{25} - \frac{9}{25} = \frac{7}{25}.$$

Answer: $\cos x = \frac{4}{5}$, $\cos(2x) = \frac{7}{25}$.

25. Solve $\cos(x) = -1$ for $x \in [0, 2\pi)$.

Solution. Cosine equals -1 only at $x = \pi$ within one full period. **Answer:** $x = \pi$.