Classwork

Solutions

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Each exercise is marked with its level. In the solutions k denotes an arbitrary integer, $k \in \mathbb{Z}$.

Exercise 1 (Polynomial gcd). **Level: medium**

Compute, with the Euclidean algorithm, the gcd of

$$A(x) = x^3 - x^2 - 4x + 4 \qquad \textit{and} \qquad B(x) = x^3 - 3x^2 + 2x$$

First division, A by B:

$$\begin{array}{rl} x^3 - x^2 - 4x + 4 & \big| \; x^3 - 3x^2 + 2x \\ \cline{2-2} -x^3 + 3x^2 - 2x & \big| \; 1 \\ \cline{1-1} 2x^2 - 6x + 4 & \end{array}$$

The quotient is 1 and the remainder is $R_1(x) = 2x^2 - 6x + 4$. Second division, B by R_1:

$$\begin{array}{rl} x^3 - 3x^2 + 2x & \big| \; 2x^2 - 6x + 4 \\ \cline{2-2} -x^3 + 3x^2 - 2x & \big| \; \frac{1}{2}x \\ \cline{1-1} 0 & \end{array}$$

The remainder is 0, so the gcd is the last nonzero remainder, $2x^2 - 6x + 4$, or, choosing the monic one, $x^2 - 3x + 2$.

Check by factoring: $A(x) = x^2(x-1) - 4(x-1) = (x-1)(x-2)(x+2)$ and $B(x) = x(x-1)(x-2)$, so the gcd is $(x-1)(x-2) = x^2 - 3x + 2$.

Answer: $\gcd(A, B) = x^2 - 3x + 2$.

Exercise 2 (Polynomial equations). **Level: medium**

Solve $x^3 - 2x^2 - 5x + 6 = 0$.

The possible rational roots are the divisors of 6: $\pm 1, \pm 2, \pm 3, \pm 6$. Since $P(1) = 0$, the polynomial $P(x)$ is divisible by $x - 1$ (factor theorem). Dividing:

$$\begin{array}{rl} x^3 - 2x^2 - 5x + 6 & \big| \; x - 1 \\ \cline{2-2} -x^3 + x^2 & \big| \; x^2 - x - 6 \\ \cline{1-1} -x^2 - 5x & \\ -x^2 + x & \\ \cline{1-1} -6x + 6 & \\ -6x + 6 & \\ \cline{1-1} 0 & \end{array}$$

so $P(x) = (x-1)(x^2 - x - 6)$. Then $x^2 - x - 6 = 0 \iff (x-3)(x+2) = 0$.

Answer: $x = 1$, $x = 3$, $x = -2$.

Exercise 3 (Parabolas). **Level: easy**

For the parabola $y = x^2 - 4x + 3$ *find the vertex, the points where it meets the coordinate axes, and draw it.*

The parabola $y = ax^2 + bx + c$ with $a = 1$, $b = -4$, $c = 3$ opens upwards, since $a > 0$.

- Vertex: $x_V = -\frac{b}{2a} = 2$, $y_V = 4 - 8 + 3 = -1$, so $V = (2, -1)$.
- y-axis: for $x = 0$, $y = 3$, so the point is $(0, 3)$.
- x-axis: $x^2 - 4x + 3 = 0 \iff (x-1)(x-3) = 0$, so the points are $(1, 0)$ and $(3, 0)$.

With the vertex and these points we can draw the graph.

Exercise 4 (Circles). **Level: medium**

Write the equation of the circle with center $C = (1, -2)$ passing through $P = (4, 2)$, and find the other endpoint Q of the diameter through P.

The radius is the distance $CP = \sqrt{(4-1)^2 + (2+2)^2} = \sqrt{9+16} = 5$, so the circle is

$$(x-1)^2 + (y+2)^2 = 25 \iff x^2 + y^2 - 2x + 4y - 20 = 0$$

The center C is the midpoint of PQ, so $Q = 2C - P = (2 \cdot 1 - 4,\ 2 \cdot (-2) - 2) = (-2, -6)$. Check: $(-2-1)^2 + (-6+2)^2 = 9 + 16 = 25$, so Q is on the circle.

Exercise 5 (Exponential equations). **Level: medium**

Solve $4^x - 3 \cdot 2^{x+1} + 8 = 0$.

Since $4^x = (2^x)^2$ and $2^{x+1} = 2 \cdot 2^x$, with $t = 2^x > 0$ the equation becomes

$$t^2 - 6t + 8 = 0 \iff (t-2)(t-4) = 0 \iff t = 2 \text{ or } t = 4$$

Both values are positive. $2^x = 2 \iff x = 1$ and $2^x = 4 \iff x = 2$.

Answer: $x = 1$ or $x = 2$.

Exercise 6 (Logarithmic equations). **Level: medium**

Solve $\log_2(x+1) + \log_2(x-1) = 3$.

Domain: $x + 1 > 0$ and $x - 1 > 0$, that is $x > 1$. In the domain, by the property $\log_2 a + \log_2 b = \log_2(ab)$,

$$\log_2\big((x+1)(x-1)\big) = 3 \iff x^2 - 1 = 2^3 = 8 \iff x^2 = 9 \iff x = \pm 3$$

The value $x = -3$ is not in the domain.

Answer: $x = 3$.

Exercise 7 (Trigonometry). **Level: easy**

Knowing that $\sin\alpha = \frac{3}{5}$ *and* $\frac{\pi}{2} < \alpha < \pi$, *compute* $\cos\alpha$, $\tan\alpha$, $\sin 2\alpha$ *and* $\cos 2\alpha$.

From $\sin^2\alpha + \cos^2\alpha = 1$ we get $\cos^2\alpha = 1 - \frac{9}{25} = \frac{16}{25}$, so $\cos\alpha = \pm\frac{4}{5}$. Since α is in the second quadrant, $\cos\alpha < 0$:

$$\cos\alpha = -\frac{4}{5}, \qquad \tan\alpha = \frac{\sin\alpha}{\cos\alpha} = -\frac{3}{4}$$

With the duplication formulas

$$\sin 2\alpha = 2\sin\alpha\cos\alpha = 2 \cdot \frac{3}{5} \cdot \left(-\frac{4}{5}\right) = -\frac{24}{25}, \qquad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \frac{16}{25} - \frac{9}{25} = \frac{7}{25}$$

Check: $\sin^2 2\alpha + \cos^2 2\alpha = \frac{576+49}{625} = 1$.

Exercise 8 (Trigonometric equations). **Level: easy**

Solve $2\cos^2 x - 3\cos x + 1 = 0$.

With $t = \cos x$: $2t^2 - 3t + 1 = 0$, so $t = \frac{3 \pm 1}{4}$, that is $t = 1$ or $t = \frac{1}{2}$, both in $[-1, 1]$.

- $\cos x = 1 \iff x = 2k\pi$;

- $\cos x = \frac{1}{2} = \cos\frac{\pi}{3} \iff x = \pm\frac{\pi}{3} + 2k\pi$.

Answer: $x = 2k\pi$ or $x = \pm\frac{\pi}{3} + 2k\pi$.

Exercise 9 (Trigonometric equations). **Level: medium**

Solve $\sin x - \cos x = 1$ *using the tangent of the half angle.*

Here $a = 1$, $b = -1$, $c = 1$.

1. For $x = \pi$: $\sin\pi - \cos\pi = 0 - (-1) = 1$. So $x = \pi + 2k\pi$ are solutions.
2. For $x \neq \pi + 2k\pi$, with $t = \tan\frac{x}{2}$:

$$\frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2} = 1 \iff 2t - 1 + t^2 = 1 + t^2 \iff 2t = 2 \iff t = 1$$

   (the term in t^2 cancels, since $x = \pi$ is a solution).
3. $\tan\frac{x}{2} = 1 \iff \frac{x}{2} = \frac{\pi}{4} + k\pi \iff x = \frac{\pi}{2} + 2k\pi$.

Answer: $x = \frac{\pi}{2} + 2k\pi$ or $x = \pi + 2k\pi$.

Exercise 10 (Exponentials and trigonometry). **Level: VERY DIFFICULT**

Solve $2^{\sin^2 x} + 2^{\cos^2 x} = 3$.

Since $\cos^2 x = 1 - \sin^2 x$, we have $2^{\cos^2 x} = 2^{1-\sin^2 x} = \dfrac{2}{2^{\sin^2 x}}$. Set $u = 2^{\sin^2 x}$. Since $0 \leq \sin^2 x \leq 1$, we have $1 \leq u \leq 2$, and the equation becomes

$$u + \frac{2}{u} = 3 \iff u^2 - 3u + 2 = 0 \iff (u-1)(u-2) = 0 \iff u = 1 \text{ or } u = 2$$

Both values are in $[1, 2]$.

- $u = 1 \iff 2^{\sin^2 x} = 2^0 \iff \sin^2 x = 0 \iff \sin x = 0 \iff x = k\pi$;
- $u = 2 \iff 2^{\sin^2 x} = 2^1 \iff \sin^2 x = 1 \iff \sin x = \pm 1 \iff x = \frac{\pi}{2} + k\pi$.

Together, the solutions are the multiples of $\frac{\pi}{2}$. Check for $x = 0$: $2^0 + 2^1 = 3$; for $x = \frac{\pi}{2}$: $2^1 + 2^0 = 3$.

Answer: $x = \frac{k\pi}{2}$.

Exercise 11 (Polynomial equations). **Level: BEYOND HIGH SCHOOL (graduate)**

Show that the equation $x^5 - x - 1 = 0$ *has exactly one real solution, and prove that this solution cannot be written starting from the integers using only* $+$, $-$, $\times$, $\div$ *and* n*-th roots (that is, the equation is not solvable by radicals).*

Let $f(x) = x^5 - x - 1$.

One real root. $f'(x) = 5x^4 - 1$ vanishes at $\pm a$, $a = 5^{-1/4} \approx 0.67$. The local maximum is at $-a$, and $f(-a) = -a^5 + a - 1 = a\left(1 - \frac{1}{5}\right) - 1 = \frac{4a}{5} - 1 < 0$. So f is negative on $(-\infty, a]$, increasing on $[a, +\infty)$, and $f(1) = -1 < 0 < f(2) = 29$: there is exactly one real root, $x_0 \approx 1.1673$, in $(1, 2)$. The other four roots are non-real, two pairs of complex conjugates.

Not solvable by radicals. By Galois' theorem, a polynomial with rational coefficients is solvable by radicals if and only if its Galois group G (the group of permutations of the five complex roots induced by the automorphisms of the splitting field) is a solvable group. We show that $G = S_5$, which is not solvable (its only proper nontrivial normal subgroup is A_5, which is simple and non-abelian).

1. *Discriminant.* For $x^5 + px + q$ the discriminant is $5^5 q^4 + 4^4 p^5$; here $p = q = -1$, so $\Delta = 3125 - 256 = 2869 = 19 \cdot 151$. Since $2 \nmid \Delta$ and $5 \nmid \Delta$, f has no repeated roots modulo 2 and 5, and by Dedekind's theorem the factorization of f modulo such a prime gives the cycle type of an element of G (a Frobenius element).

2. *Modulo* 5. For every $c \in \mathbb{F}_5$, $c^5 = c$ (Fermat), so $f(c) = -1 \neq 0$: f has no roots in $\mathbb{F}_5$. Polynomials of the form $x^p - x - b$ with $b \neq 0$ are irreducible over $\mathbb{F}_p$ (Artin–Schreier), so f is irreducible modulo 5. Hence f is irreducible over $\mathbb{Q}$, and G contains a 5-cycle.

3. *Modulo* 2. $f \equiv x^5 + x + 1 = (x^2 + x + 1)(x^3 + x^2 + 1)$, and both factors are irreducible over $\mathbb{F}_2$ (they have no roots, and have degree at most 3). So G contains an element σ of cycle type $(2, 3)$, and σ^3 is a transposition.

4. *Conclusion.* A subgroup of S_5 containing a 5-cycle and a transposition is all of S_5 (this holds in S_p for every prime p). So $G = S_5$, which is not solvable.

Therefore x_0, although it exists and can be computed numerically to any precision, cannot be expressed by radicals. (Abel–Ruffini: there is no general formula with radicals for equations of degree 5; this exercise exhibits a specific one.)