Math0 First lesson Exercises 3

FCS UNIPI – Math (Caboara)

Version A

1. Is it true that $\sqrt{7}+1+\sqrt{3}>2+\sqrt{6}$?

2. Solve, considering existence conditions, the equation
$$\frac{\sqrt{x^2+9}}{x^2}=0.$$

3. Simplify
$$\frac{m^2-n^2}{m-n}+5.$$

12. Draw on the Cartesian plane the parabola $f(x)=x^2-4x+3$.

17. Put in rational standard form
$$\frac{\sqrt{2}}{3-\sqrt{2}}.$$

Version B

1. Is it true that $\sqrt{10}+2+\sqrt{3}>3+\sqrt{8}$?

2. Solve, considering existence conditions, the equation
$$\frac{\sqrt{x^2+16}}{x^2}=0.$$

3. Simplify
$$\frac{a^2-b^2}{a-b}-2.$$

12. Draw on the Cartesian plane the parabola $f(x)=x^2+2x-3$.

17. Put in rational standard form
$$\frac{\sqrt{5}}{3-\sqrt{5}}.$$

Version C

1. Is it true that $\sqrt{6}+1+\sqrt{5} > 2+\sqrt{7}$?
2. Solve, considering existence conditions, the equation
$$\frac{\sqrt{x^2+25}}{x^2} = 0.$$
3. Simplify
$$\frac{x^2-y^2}{x-y} + 7.$$

12. Draw on the Cartesian plane the parabola $f(x) = x^2 - 2x - 8$.

17. Put in rational standard form
$$\frac{\sqrt{2}}{5-\sqrt{2}}.$$

Version D

1. Is it true that $\sqrt{11}+1+\sqrt{2} > 3+\sqrt{5}$?
2. Solve, considering existence conditions, the equation
$$\frac{\sqrt{x^2+36}}{x^2} = 0.$$
3. Simplify
$$\frac{u^2-v^2}{u-v} - 4.$$

12. Draw on the Cartesian plane the parabola $f(x) = x^2 + 4x - 5$.

17. Put in rational standard form
$$\frac{\sqrt{3}}{5-\sqrt{3}}.$$

Version E

1. Is it true that $\sqrt{13} + 2 + \sqrt{3} > 4 + \sqrt{7}$?

2. Solve, considering existence conditions, the equation
$$\frac{\sqrt{x^2 + 49}}{x^2} = 0.$$

3. Simplify
$$\frac{r^2 - s^2}{r - s} + 1.$$

12. Draw on the Cartesian plane the parabola $f(x) = x^2 - 6x + 5$.

17. Put in rational standard form
$$\frac{\sqrt{5}}{4 - \sqrt{5}}.$$

Step-by-step solutions

The five versions have the same structure, so the methods are the same in every version:

- **Ex. 1:** compare without a calculator, term by term, by integer bounds, or by squaring positive quantities.
- **Ex. 2:** a fraction is defined only if its denominator is $\neq 0$, and $\sqrt{A}$ only if $A \geq 0$; a fraction equals 0 exactly when its numerator is 0.
- **Ex. 3:** use $A^2 - B^2 = (A-B)(A+B)$; the existence condition stays valid after simplifying.
- **Ex. 12:** for $f(x) = x^2 + bx + c$ find the roots, the vertex $V = \left(-\frac{b}{2}, f\left(-\frac{b}{2}\right)\right)$, the y-intercept $(0, c)$ and its symmetric point with respect to the axis $x = -\frac{b}{2}$.
- **Ex. 17:** multiply numerator and denominator by the conjugate of the denominator, using $(A - B)(A+B) = A^2 - B^2$.

Version A

Exercise 1. We compare the two sides term by term.

- Since $7 > 6$, $\sqrt{7} > \sqrt{6}$.
- Since $3 > 1$, $\sqrt{3} > \sqrt{1} = 1$, hence $1 + \sqrt{3} > 1 + 1 = 2$.

Adding the two inequalities: $\sqrt{7} + 1 + \sqrt{3} > \sqrt{6} + 2$.
Answer: yes, the inequality is true.

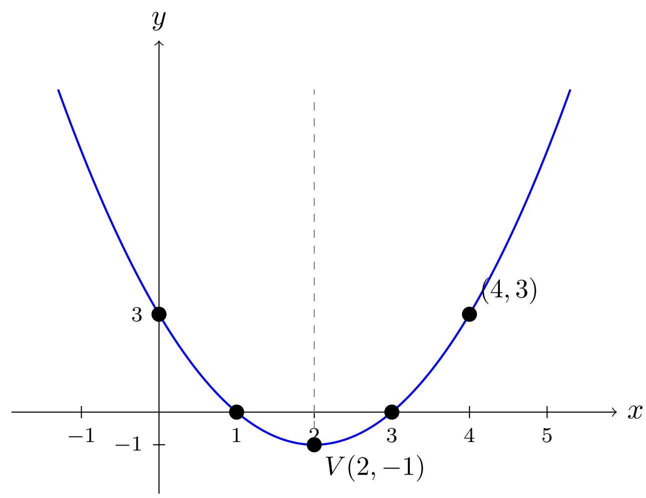
Exercise 2. *Existence conditions:* $x^2 \neq 0 \iff x \neq 0$; $x^2 + 9 \geq 9 > 0$ for every x, so the square root is always defined.
Equation: $\sqrt{x^2+9} = 0 \iff x^2 + 9 = 0 \iff x^2 = -9$, which has no real solutions. (Indeed the numerator is always $\geq \sqrt{9} = 3 > 0$.)
Answer: $S = \emptyset$.

Exercise 3. Condition: $m - n \neq 0 \iff m \neq n$. Using $m^2 - n^2 = (m-n)(m+n)$:

$$\frac{m^2 - n^2}{m - n} + 5 = \frac{(m-n)(m+n)}{m-n} + 5 = m + n + 5, \qquad m \neq n.$$

Exercise 12. $f(x) = x^2 - 4x + 3$, with $a = 1 > 0$: the parabola opens upwards.

- Roots: we look for two numbers with sum 4 and product 3, namely 1 and 3: $x^2 - 4x + 3 = (x-1)(x-3) = 0 \iff x = 1$ or $x = 3$.
- Vertex: $x_V = -\dfrac{-4}{2} = 2$ (the midpoint of the roots), $f(2) = 4 - 8 + 3 = -1$, so $V = (2, -1)$. Axis of symmetry: $x = 2$.
- y-intercept: $f(0) = 3$, so $(0, 3)$; its symmetric point w.r.t. $x = 2$ is $(4, 3)$.



Exercise 17. Multiply numerator and denominator by the conjugate $3+\sqrt{2}$:

$$\frac{\sqrt{2}}{3-\sqrt{2}}\cdot\frac{3+\sqrt{2}}{3+\sqrt{2}}=\frac{\sqrt{2}\,(3+\sqrt{2})}{3^2-(\sqrt{2})^2}=\frac{3\sqrt{2}+2}{9-2}=\frac{3\sqrt{2}+2}{7}.$$

Version B

Exercise 1. We compare the two sides term by term.

- Since $10>8$, $\sqrt{10}>\sqrt{8}$.
- Since $3>1$, $\sqrt{3}>1$, hence $2+\sqrt{3}>2+1=3$.

Adding the two inequalities: $\sqrt{10}+2+\sqrt{3}>\sqrt{8}+3$.
Answer: yes, the inequality is true.

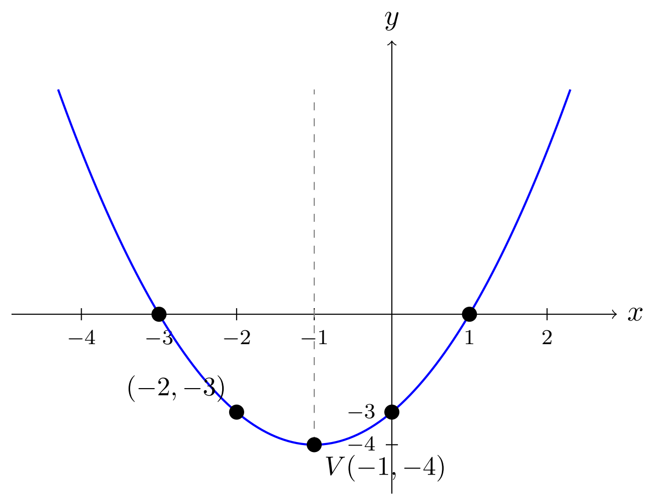
Exercise 2. *Existence conditions:* $x^2\neq 0 \iff x\neq 0$; $x^2+16\geq 16>0$ for every x, so the square root is always defined.
Equation: $\sqrt{x^2+16}=0 \iff x^2+16=0 \iff x^2=-16$, which has no real solutions. (Indeed the numerator is always $\geq\sqrt{16}=4>0$.)
Answer: $S=\emptyset$.

Exercise 3. Condition: $a-b\neq 0 \iff a\neq b$. Using $a^2-b^2=(a-b)(a+b)$:

$$\frac{a^2-b^2}{a-b}-2=\frac{(a-b)(a+b)}{a-b}-2=a+b-2,\qquad a\neq b.$$

Exercise 12. $f(x)=x^2+2x-3$, with $a=1>0$: the parabola opens upwards.

- Roots: we look for two numbers with sum -2 and product -3, namely -3 and 1: $x^2+2x-3=(x+3)(x-1)=0 \iff x=-3$ or $x=1$.
- Vertex: $x_V=-\dfrac{2}{2}=-1$ (the midpoint of the roots), $f(-1)=1-2-3=-4$, so $V=(-1,-4)$. Axis of symmetry: $x=-1$.
- y-intercept: $f(0)=-3$, so $(0,-3)$; its symmetric point w.r.t. $x=-1$ is $(-2,-3)$.



Exercise 17. Multiply numerator and denominator by the conjugate $3+\sqrt{5}$:

$$\frac{\sqrt{5}}{3-\sqrt{5}}\cdot\frac{3+\sqrt{5}}{3+\sqrt{5}}=\frac{\sqrt{5}\,(3+\sqrt{5})}{3^2-(\sqrt{5})^2}=\frac{3\sqrt{5}+5}{9-5}=\frac{3\sqrt{5}+5}{4}.$$

Version C

Exercise 1. Here the term-by-term comparison $\sqrt{6}$ vs $\sqrt{7}$ goes the wrong way ($\sqrt{6}<\sqrt{7}$), so we estimate both sides with integers instead.

- Left side: $\sqrt{6}>\sqrt{4}=2$ and $\sqrt{5}>\sqrt{4}=2$, hence $\sqrt{6}+1+\sqrt{5}>2+1+2=5$.
- Right side: $\sqrt{7}<\sqrt{9}=3$, hence $2+\sqrt{7}<2+3=5$.

Therefore $\sqrt{6}+1+\sqrt{5}>5>2+\sqrt{7}$.
Answer: yes, the inequality is true.

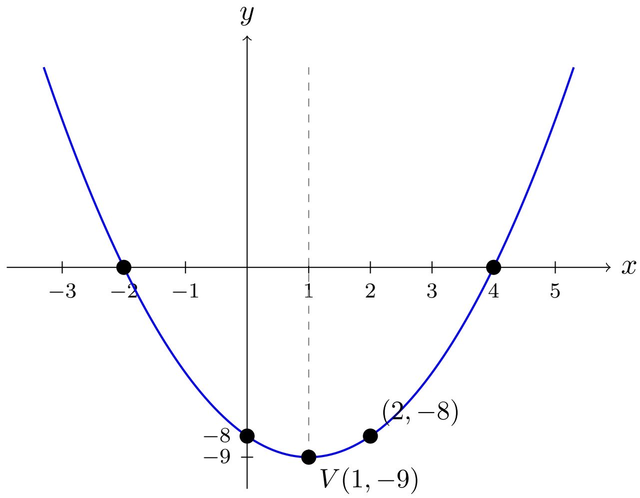
Exercise 2. *Existence conditions:* $x^2\neq 0 \iff x\neq 0$; $x^2+25\geq 25>0$ for every x, so the square root is always defined.
Equation: $\sqrt{x^2+25}=0 \iff x^2+25=0 \iff x^2=-25$, which has no real solutions. (Indeed the numerator is always $\geq\sqrt{25}=5>0$.)
Answer: $S=\emptyset$.

Exercise 3. Condition: $x-y\neq 0 \iff x\neq y$. Using $x^2-y^2=(x-y)(x+y)$:

$$\frac{x^2-y^2}{x-y}+7=\frac{(x-y)(x+y)}{x-y}+7=x+y+7,\qquad x\neq y.$$

Exercise 12. $f(x)=x^2-2x-8$, with $a=1>0$: the parabola opens upwards.

- Roots: we look for two numbers with sum 2 and product -8, namely -2 and 4: $x^2-2x-8=(x+2)(x-4)=0 \iff x=-2$ or $x=4$.
- Vertex: $x_V=-\dfrac{-2}{2}=1$ (the midpoint of the roots), $f(1)=1-2-8=-9$, so $V=(1,-9)$. Axis of symmetry: $x=1$.
- y-intercept: $f(0)=-8$, so $(0,-8)$; its symmetric point w.r.t. $x=1$ is $(2,-8)$.



Exercise 17. Multiply numerator and denominator by the conjugate $5+\sqrt{2}$:

$$\frac{\sqrt{2}}{5-\sqrt{2}}\cdot\frac{5+\sqrt{2}}{5+\sqrt{2}}=\frac{\sqrt{2}\,(5+\sqrt{2})}{5^2-(\sqrt{2})^2}=\frac{5\sqrt{2}+2}{25-2}=\frac{5\sqrt{2}+2}{23}.$$

Version D

Exercise 1. We pair the terms suitably.

- Since $11>9$, $\sqrt{11}>\sqrt{9}=3$.
- $1+\sqrt{2}$ vs $\sqrt{5}$: both are positive, so we compare squares: $(1+\sqrt{2})^2=3+2\sqrt{2}$ and $(\sqrt{5})^2=5$. Now $3+2\sqrt{2}>5 \iff 2\sqrt{2}>2 \iff \sqrt{2}>1$, which is true. Hence $1+\sqrt{2}>\sqrt{5}$.

Adding the two inequalities: $\sqrt{11}+1+\sqrt{2}>3+\sqrt{5}$.
Answer: yes, the inequality is true.

Exercise 2. *Existence conditions:* $x^2\neq 0 \iff x\neq 0$; $x^2+36\geq 36>0$ for every x, so the square root is always defined.
Equation: $\sqrt{x^2+36}=0 \iff x^2+36=0 \iff x^2=-36$, which has no real solutions. (Indeed the numerator is always $\geq\sqrt{36}=6>0$.)
Answer: $S=\emptyset$.

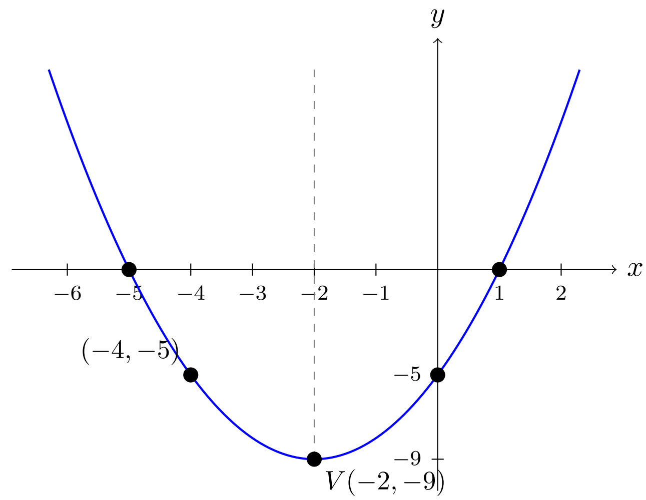
Exercise 3. Condition: $u-v\neq 0 \iff u\neq v$. Using $u^2-v^2=(u-v)(u+v)$:

$$\frac{u^2-v^2}{u-v}-4=\frac{(u-v)(u+v)}{u-v}-4=u+v-4,\qquad u\neq v.$$

Exercise 12. $f(x)=x^2+4x-5$, with $a=1>0$: the parabola opens upwards.

- Roots: we look for two numbers with sum -4 and product -5, namely -5 and 1: $x^2+4x-5=(x+5)(x-1)=0 \iff x=-5$ or $x=1$.
- Vertex: $x_V=-\dfrac{4}{2}=-2$ (the midpoint of the roots), $f(-2)=4-8-5=-9$, so $V=(-2,-9)$. Axis of symmetry: $x=-2$.

- y-intercept: $f(0) = -5$, so $(0, -5)$; its symmetric point w.r.t. $x = -2$ is $(-4, -5)$.



Exercise 17. Multiply numerator and denominator by the conjugate $5 + \sqrt{3}$:

$$\frac{\sqrt{3}}{5-\sqrt{3}} \cdot \frac{5+\sqrt{3}}{5+\sqrt{3}} = \frac{\sqrt{3}\,(5+\sqrt{3})}{5^2-(\sqrt{3})^2} = \frac{5\sqrt{3}+3}{25-3} = \frac{5\sqrt{3}+3}{22}.$$

Version E

Exercise 1. Subtract 2 from both sides: the inequality is equivalent to

$$\sqrt{13}+\sqrt{3} > 2+\sqrt{7}.$$

Both sides are positive, so we compare their squares:

$$(\sqrt{13}+\sqrt{3})^2 = 16+2\sqrt{39} = 16+\sqrt{156}, \qquad (2+\sqrt{7})^2 = 4+4\sqrt{7}+7 = 11+\sqrt{112}.$$

Since $16 > 11$ and $\sqrt{156} > \sqrt{112}$, the left square is larger, hence $\sqrt{13}+\sqrt{3} > 2+\sqrt{7}$.
(A term-by-term comparison does not work here: $\sqrt{13} > \sqrt{7}$, but $2+\sqrt{3} < 4$.)
Answer: yes, the inequality is true.

Exercise 2. *Existence conditions:* $x^2 \neq 0 \iff x \neq 0$; $x^2+49 \geq 49 > 0$ for every x, so the square root is always defined.
Equation: $\sqrt{x^2+49} = 0 \iff x^2+49 = 0 \iff x^2 = -49$, which has no real solutions. (Indeed the numerator is always $\geq \sqrt{49} = 7 > 0$.)
Answer: $S = \emptyset$.

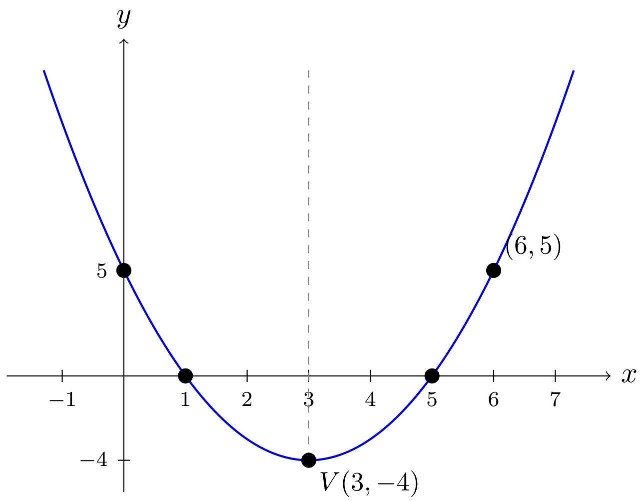
Exercise 3. Condition: $r - s \neq 0 \iff r \neq s$. Using $r^2 - s^2 = (r-s)(r+s)$:

$$\frac{r^2-s^2}{r-s}+1 = \frac{(r-s)(r+s)}{r-s}+1 = r+s+1, \qquad r \neq s.$$

Exercise 12. $f(x) = x^2 - 6x + 5$, with $a = 1 > 0$: the parabola opens upwards.

- Roots: we look for two numbers with sum 6 and product 5, namely 1 and 5: $x^2 - 6x + 5 = (x-1)(x-5) = 0 \iff x = 1$ or $x = 5$.

- Vertex: $x_V = -\dfrac{-6}{2} = 3$ (the midpoint of the roots), $f(3) = 9 - 18 + 5 = -4$, so $V = (3, -4)$. Axis of symmetry: $x = 3$.
- y-intercept: $f(0) = 5$, so $(0, 5)$; its symmetric point w.r.t. $x = 3$ is $(6, 5)$.



Exercise 17. Multiply numerator and denominator by the conjugate $4 + \sqrt{5}$:

$$\frac{\sqrt{5}}{4 - \sqrt{5}} \cdot \frac{4 + \sqrt{5}}{4 + \sqrt{5}} = \frac{\sqrt{5}\,(4 + \sqrt{5})}{4^2 - (\sqrt{5})^2} = \frac{4\sqrt{5} + 5}{16 - 5} = \frac{4\sqrt{5} + 5}{11}.$$