Math0 First Lesson Exercises 2

FCS UNIPI – Math (Caboara)

Topic 1 – Comparing / estimating sums of irrational numbers

1. Is it true that $\sqrt{8} + 1 + \sqrt{5} > 3 + \sqrt{6}$?
2. Without using a calculator, order the following three numbers from smallest to largest:
$$\sqrt{7}, \qquad 1 + \sqrt{2}, \qquad 3.$$
3. Without using a calculator, determine which of the two numbers is larger:
$$\sqrt{11} + \sqrt{2} \qquad \text{or} \qquad \sqrt{26}.$$

Topic 2 – Equations with a square root over a polynomial, existence conditions

4. Solve, considering existence conditions, the equation
$$\frac{\sqrt{x^2 + 20}}{x^2} = 0.$$
5. Solve, considering existence conditions, the equation
$$\frac{\sqrt{x^2 - 9}}{x - 4} = 0.$$

Topic 3 – Simplifying a difference-of-squares expression

6. Simplify
$$\frac{p^2 - q^2}{p - q} + 4.$$
7. Simplify
$$\frac{p^2 - q^2}{p + q} - 5.$$
8. Simplify
$$\frac{(p + q)^2 - (p - q)^2}{4q}.$$

Topic 4 – Graphing a parabola

9. Draw on the Cartesian plane the parabola $f(x) = x^2 - 2x - 15$.
10. A parabola has roots $x = -2$ and $x = 3$, and passes through the point $(0, -12)$. Find its equation and draw it on the Cartesian plane.

Topic 5 – Rationalizing a denominator

11. Put in rational standard form
$$\frac{\sqrt{7}}{5-\sqrt{7}}.$$
12. Put in rational standard form
$$\frac{1}{\sqrt{5}+\sqrt{2}}.$$

Step-by-step solutions

Topic 1 – Comparing / estimating sums of irrational numbers

Exercise 1. We compare the two sides term by term.

- Since $8 > 6$ and the square root is increasing, $\sqrt{8} > \sqrt{6}$.
- Since $5 > 4$, we have $\sqrt{5} > \sqrt{4} = 2$, hence $1 + \sqrt{5} > 1 + 2 = 3$.

Adding the two inequalities:

$$\sqrt{8} + 1 + \sqrt{5} > \sqrt{6} + 3.$$

Answer: yes, the inequality is true.

Exercise 2.

- $\sqrt{7}$ vs 3: since $7 < 9$, $\sqrt{7} < \sqrt{9} = 3$.
- $1 + \sqrt{2}$ vs $\sqrt{7}$: both are positive, so we compare their squares:

  $$(1 + \sqrt{2})^2 = 1 + 2\sqrt{2} + 2 = 3 + 2\sqrt{2}, \qquad (\sqrt{7})^2 = 7.$$

  Now $3 + 2\sqrt{2} < 7 \iff 2\sqrt{2} < 4$, and squaring again (both sides positive) $\iff 8 < 16$, which is true. Hence $1 + \sqrt{2} < \sqrt{7}$.

Answer: $1 + \sqrt{2} < \sqrt{7} < 3$.

Exercise 3. Both numbers are positive, so we compare their squares:

$$(\sqrt{11} + \sqrt{2})^2 = 11 + 2\sqrt{22} + 2 = 13 + 2\sqrt{22}, \qquad (\sqrt{26})^2 = 26.$$

Since $16 < 22 < 25$, we have $4 < \sqrt{22} < 5$, hence $2\sqrt{22} < 10$ and

$$13 + 2\sqrt{22} < 13 + 10 = 23 < 26.$$

(Alternatively: $13 + 2\sqrt{22} < 26 \iff 2\sqrt{22} < 13 \iff 88 < 169$.)
Answer: $\sqrt{26} > \sqrt{11} + \sqrt{2}$.

Topic 2 – Equations with a square root over a polynomial, existence conditions

We use two facts: a fraction is defined only when its denominator is nonzero, and $\sqrt{A}$ is defined only when $A \geq 0$; a fraction equals 0 exactly when its numerator is 0 (and the fraction is defined).

Exercise 4. *Existence conditions:* $x^2 \neq 0 \iff x \neq 0$; $x^2 + 20 \geq 20 > 0$ for every x.
Equation: $\sqrt{x^2 + 20} = 0 \iff x^2 + 20 = 0 \iff x^2 = -20$, which has no real solutions.
Answer: $S = \emptyset$.

Exercise 5. *Existence conditions:* $x - 4 \neq 0 \iff x \neq 4$; $x^2 - 9 \geq 0 \iff x^2 \geq 9 \iff x \leq -3$ or $x \geq 3$. So the domain is $(-\infty, -3] \cup [3, 4) \cup (4, +\infty)$.
Equation: $\sqrt{x^2 - 9} = 0 \iff x^2 = 9 \iff x = \pm 3$. Both values belong to the domain (in particular, neither equals 4).
Answer: $S = \{-3, 3\}$.

Topic 3 – Simplifying a difference-of-squares expression

Key identity: $p^2 - q^2 = (p-q)(p+q)$. The existence condition (denominator $\neq 0$) remains valid for the simplified expression.

Exercise 6. Condition: $p - q \neq 0 \iff p \neq q$.

$$\frac{p^2 - q^2}{p-q} + 4 = \frac{(p-q)(p+q)}{p-q} + 4 = p + q + 4, \qquad p \neq q.$$

Exercise 7. Condition: $p + q \neq 0 \iff p \neq -q$.

$$\frac{p^2 - q^2}{p+q} - 5 = \frac{(p-q)(p+q)}{p+q} - 5 = p - q - 5, \qquad p \neq -q.$$

Exercise 8. Condition: $4q \neq 0 \iff q \neq 0$. The numerator is a difference of squares $A^2 - B^2 = (A-B)(A+B)$ with $A = p+q$, $B = p-q$:

$$A - B = (p+q) - (p-q) = 2q, \qquad A + B = (p+q) + (p-q) = 2p.$$

Hence

$$\frac{(p+q)^2 - (p-q)^2}{4q} = \frac{(2q)(2p)}{4q} = \frac{4pq}{4q} = p, \qquad q \neq 0.$$

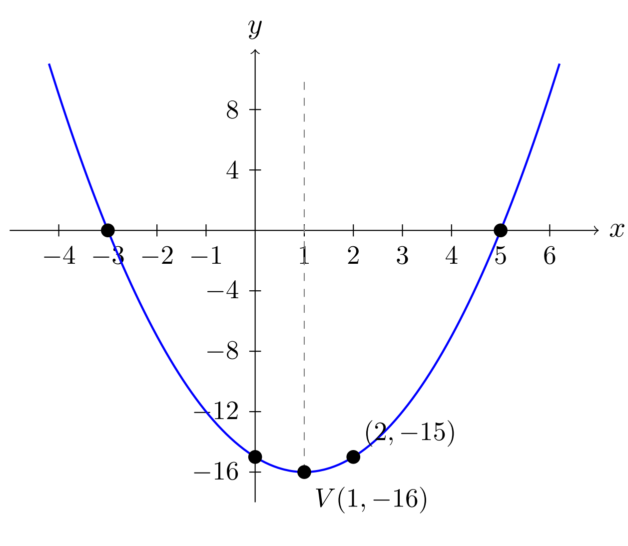
(Alternatively, expand: $(p^2 + 2pq + q^2) - (p^2 - 2pq + q^2) = 4pq$.)

Topic 4 – Graphing a parabola

For $f(x) = ax^2 + bx + c$ we find: the direction of opening (upwards if $a > 0$), the roots, the vertex $V = \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$, the y-intercept $(0, c)$, and its symmetric point with respect to the axis $x = -\frac{b}{2a}$.

Exercise 9. $f(x) = x^2 - 2x - 15$, with $a = 1 > 0$: the parabola opens upwards.

- Roots: we look for two numbers with sum -2 and product -15, namely -5 and 3: $x^2 - 2x - 15 = (x-5)(x+3) = 0 \iff x = 5$ or $x = -3$.
- Vertex: $x_V = -\dfrac{-2}{2} = 1$ (the midpoint of the roots), $f(1) = 1 - 2 - 15 = -16$, so $V = (1, -16)$. Axis of symmetry: $x = 1$.
- y-intercept: $f(0) = -15$, so $(0, -15)$; its symmetric point w.r.t. $x = 1$ is $(2, -15)$.



Exercise 10. A parabola with roots $x = -2$ and $x = 3$ has the form

$$f(x) = a(x+2)(x-3), \qquad a \neq 0.$$

Imposing that it passes through $(0, -12)$:

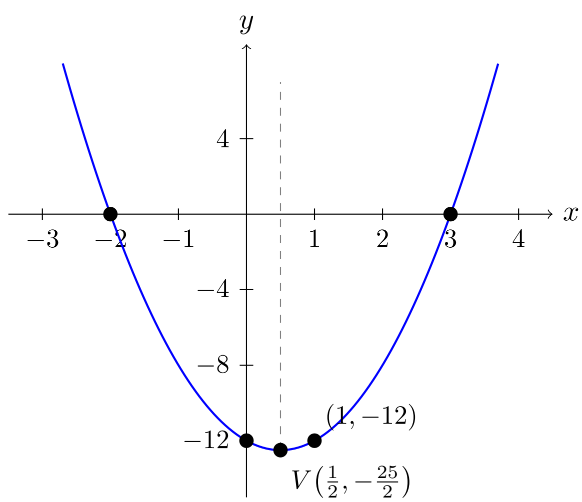
$$f(0) = a(0+2)(0-3) = -6a = -12 \iff a = 2.$$

Hence

$$f(x) = 2(x+2)(x-3) = 2(x^2 - x - 6) = 2x^2 - 2x - 12.$$

Since $a = 2 > 0$, the parabola opens upwards.

- Vertex: $x_V = -\dfrac{-2}{2 \cdot 2} = \dfrac{1}{2}$ (the midpoint of -2 and 3), $f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}+2\right)\left(\frac{1}{2}-3\right) = 2 \cdot \frac{5}{2} \cdot \left(-\frac{5}{2}\right) = -\frac{25}{2}$, so $V = \left(\frac{1}{2}, -\frac{25}{2}\right)$. Axis: $x = \frac{1}{2}$.
- y-intercept: $(0, -12)$; its symmetric point w.r.t. $x = \frac{1}{2}$ is $(1, -12)$.



Topic 5 – Rationalizing a denominator

Key identity: $(A - B)(A + B) = A^2 - B^2$. Multiplying by the *conjugate* eliminates the square roots from the denominator.

Exercise 11. Multiply numerator and denominator by the conjugate $5 + \sqrt{7}$:

$$\frac{\sqrt{7}}{5 - \sqrt{7}} \cdot \frac{5 + \sqrt{7}}{5 + \sqrt{7}} = \frac{5\sqrt{7} + 7}{25 - 7} = \frac{7 + 5\sqrt{7}}{18}.$$

Exercise 12. Multiply by the conjugate $\sqrt{5} - \sqrt{2}$:

$$\frac{1}{\sqrt{5} + \sqrt{2}} \cdot \frac{\sqrt{5} - \sqrt{2}}{\sqrt{5} - \sqrt{2}} = \frac{\sqrt{5} - \sqrt{2}}{5 - 2} = \frac{\sqrt{5} - \sqrt{2}}{3}.$$