Math 0 First Lesson Exercises 1

FCS UNIPI – Math (Caboara)

Topic 1 – Comparing / estimating sums of irrational numbers

1. Is it true that $\sqrt{12} + 2 + \sqrt{7} > 4 + \sqrt{10}$?
2. Is it true that $\sqrt{10} + 2 > 4 + \sqrt{3}$?
3. Without using a calculator, order the following three numbers from smallest to largest:
$$\sqrt{13}, \qquad 2 + \sqrt{2}, \qquad 4.$$
4. Without using a calculator, determine which of the two numbers is larger:
$$\sqrt{7} + \sqrt{3} \qquad \text{or} \qquad \sqrt{17}.$$
5. Without using a calculator, determine between which two consecutive integers the number $\sqrt{5} + \sqrt{11}$ lies.
6. Is it true that $3 + \sqrt{2} > \sqrt{5} + \sqrt{3}$?

Topic 2 – Equations with a square root over a polynomial, existence conditions

7. Solve, considering existence conditions, the equation
$$\frac{\sqrt{x^2 + 30}}{x^2} = 0.$$
8. Solve, considering existence conditions, the equation
$$\frac{\sqrt{9 - x^2}}{x} = 0.$$
9. Solve, considering existence conditions, the equation
$$\frac{\sqrt{x^2 + 5}}{x^2 - 4} = 0.$$
10. Solve, considering existence conditions, the equation
$$\frac{\sqrt{16 - x^2}}{x - 1} = 0.$$

Topic 3 – Simplifying an algebraic expression

11. Simplify
$$\frac{a^2-9}{a-3}+2.$$

12. Simplify
$$\frac{16-b^2}{4-b}-3.$$

13. Simplify
$$\frac{x^3-x}{x^2-1}.$$

14. Simplify
$$\frac{(2p-q)^2-(2p+q)^2}{q}.$$

15. Simplify
$$\frac{m^2-4mn+4n^2}{m-2n}.$$

16. Simplify
$$\frac{25-y^2}{y+5}+y.$$

Topic 4 – Draw the Graph of a parabola

17. Draw on the Cartesian plane the parabola $f(x)=x^2-6x+8$.

18. Draw on the Cartesian plane the parabola $f(x)=(x+2)^2-16$ (first write it in the form x^2+bx+c).

19. A parabola has roots $x=-1$ and $x=5$, and passes through the point $(0,-5)$. Find its equation and draw it on the Cartesian plane.

20. A parabola $f(x)=x^2+bx+c$ satisfies $f(0)=21$ and has roots $x=3$ and $x=7$. Find $f(x)$ and draw it on the Cartesian plane.

Topic 5 – Rationalizing

21. Put in rational standard form
$$\frac{\sqrt{6}}{4-\sqrt{6}}.$$

22. Rationalize the numerator of
$$\frac{\sqrt{7}-\sqrt{5}}{2}.$$

23. Put in rational standard form
$$\frac{3}{\sqrt{7}-\sqrt{2}}.$$

24. Put in rational standard form
$$\frac{2+\sqrt{3}}{2-\sqrt{3}}.$$

Solutions

Topic 1 – Comparing / estimating sums of irrational numbers

Exercise 1. We compare the two sides term by term.

- Since $12 > 10$ and the square root is increasing, $\sqrt{12} > \sqrt{10}$.
- Since $7 > 4$, we have $\sqrt{7} > \sqrt{4} = 2$, hence $2 + \sqrt{7} > 2 + 2 = 4$.

Adding the two inequalities:
$$\sqrt{12} + 2 + \sqrt{7} > \sqrt{10} + 4.$$

Answer: yes, the inequality is true.

Exercise 2. Subtract 2 from both sides: the inequality is equivalent to

$$\sqrt{10} > 2 + \sqrt{3}.$$

Both sides are positive, so we may square without changing the direction of the inequality:

$$10 > (2+\sqrt{3})^2 = 4 + 4\sqrt{3} + 3 = 7 + 4\sqrt{3} \iff 3 > 4\sqrt{3}.$$

But $\sqrt{3} > 1$, so $4\sqrt{3} > 4 > 3$: the last inequality is false.
Answer: no, the inequality is false (in fact $\sqrt{10} + 2 < 4 + \sqrt{3}$).

Exercise 3.

- $\sqrt{13}$ vs 4: since $13 < 16$, $\sqrt{13} < \sqrt{16} = 4$.
- $2 + \sqrt{2}$ vs $\sqrt{13}$: both are positive, so we compare their squares:

  $$(2+\sqrt{2})^2 = 4 + 4\sqrt{2} + 2 = 6 + 4\sqrt{2}, \qquad (\sqrt{13})^2 = 13.$$

  Now $6 + 4\sqrt{2} < 13 \iff 4\sqrt{2} < 7$, and squaring again (both sides positive) $\iff 32 < 49$, which is true. Hence $2 + \sqrt{2} < \sqrt{13}$.

Answer: $2 + \sqrt{2} < \sqrt{13} < 4$.

Exercise 4. Both numbers are positive, so we compare their squares:

$$(\sqrt{7}+\sqrt{3})^2 = 7 + 2\sqrt{21} + 3 = 10 + 2\sqrt{21}, \qquad (\sqrt{17})^2 = 17.$$

Now $10 + 2\sqrt{21} > 17 \iff 2\sqrt{21} > 7$, and squaring (both sides positive) $\iff 84 > 49$, which is true.
Answer: $\sqrt{7} + \sqrt{3} > \sqrt{17}$.

Exercise 5. Let $N = \sqrt{5} + \sqrt{11} > 0$. Then

$$N^2 = 5 + 2\sqrt{55} + 11 = 16 + 2\sqrt{55}.$$

Since $49 < 55 < 64$, we have $7 < \sqrt{55} < 8$, hence

$$16 + 14 < N^2 < 16 + 16, \qquad \text{i.e.} \qquad 30 < N^2 < 32.$$

As $25 < 30$ and $32 < 36$, we get $25 < N^2 < 36$, and since $N > 0$,

$$5 < N < 6.$$

Answer: $\sqrt{5}+\sqrt{11}$ lies between 5 and 6.

Exercise 6. Both sides are positive, so we compare their squares:

$$(3+\sqrt{2})^2 = 9+6\sqrt{2}+2 = 11+6\sqrt{2}, \qquad (\sqrt{5}+\sqrt{3})^2 = 5+2\sqrt{15}+3 = 8+2\sqrt{15}.$$

We have $11 > 8$, and $6\sqrt{2} = \sqrt{36\cdot 2} = \sqrt{72} > \sqrt{60} = \sqrt{4\cdot 15} = 2\sqrt{15}$. Adding,

$$11+6\sqrt{2} > 8+2\sqrt{15}.$$

Answer: yes, $3+\sqrt{2} > \sqrt{5}+\sqrt{3}$.

Topic 2 – Equations with a square root over a polynomial, existence conditions

In all these exercises we use the same two facts: a fraction is defined only when its denominator is nonzero, and $\sqrt{A}$ is defined only when $A \geq 0$; a fraction is equal to 0 exactly when its numerator is 0 (and the fraction is defined).

Exercise 7. *Existence conditions:* $x^2 \neq 0 \iff x \neq 0$; $x^2+30 \geq 0$, which holds for every x since $x^2+30 \geq 30 > 0$.
Equation: $\sqrt{x^2+30} = 0 \iff x^2+30 = 0 \iff x^2 = -30$, which has no real solutions.
Answer: $S = \emptyset$.

Exercise 8. *Existence conditions:* $x \neq 0$; $9-x^2 \geq 0 \iff x^2 \leq 9 \iff -3 \leq x \leq 3$. So the domain is $[-3,0)\cup(0,3]$.
Equation: $\sqrt{9-x^2} = 0 \iff 9-x^2 = 0 \iff x = \pm 3$. Both values belong to the domain.
Answer: $S = \{-3,\, 3\}$.

Exercise 9. *Existence conditions:* $x^2-4 \neq 0 \iff x \neq \pm 2$; $x^2+5 \geq 5 > 0$ for every x.
Equation: $\sqrt{x^2+5} = 0 \iff x^2 = -5$, which has no real solutions.
Answer: $S = \emptyset$.

Exercise 10. *Existence conditions:* $x-1 \neq 0 \iff x \neq 1$; $16-x^2 \geq 0 \iff -4 \leq x \leq 4$. So the domain is $[-4,1)\cup(1,4]$.
Equation: $\sqrt{16-x^2} = 0 \iff x^2 = 16 \iff x = \pm 4$. Both values belong to the domain.
Answer: $S = \{-4,\, 4\}$.

Topic 3 – Simplifying an algebraic expression

In each case we first state the existence condition (denominator $\neq 0$), which remains valid for the simplified expression.

Exercise 11. Condition: $a \neq 3$. Using $a^2-9 = (a-3)(a+3)$:

$$\frac{a^2-9}{a-3}+2 = \frac{(a-3)(a+3)}{a-3}+2 = a+3+2 = a+5, \qquad a \neq 3.$$

Exercise 12. Condition: $b \neq 4$. Using $16-b^2 = (4-b)(4+b)$:

$$\frac{16-b^2}{4-b}-3 = \frac{(4-b)(4+b)}{4-b}-3 = 4+b-3 = b+1, \qquad b \neq 4.$$

Exercise 13. Condition: $x^2-1 \neq 0 \iff x \neq \pm 1$. Collecting x in the numerator:

$$\frac{x^3-x}{x^2-1} = \frac{x(x^2-1)}{x^2-1} = x, \qquad x \neq \pm 1.$$

Exercise 14. Condition: $q \neq 0$. The numerator is a difference of squares $A^2 - B^2 = (A-B)(A+B)$ with $A = 2p - q$, $B = 2p + q$:

$$A - B = (2p-q) - (2p+q) = -2q, \qquad A + B = (2p-q) + (2p+q) = 4p.$$

Hence

$$\frac{(2p-q)^2 - (2p+q)^2}{q} = \frac{(-2q)(4p)}{q} = \frac{-8pq}{q} = -8p, \qquad q \neq 0.$$

(Alternatively, expand: $(4p^2 - 4pq + q^2) - (4p^2 + 4pq + q^2) = -8pq$.)

Exercise 15. Condition: $m \neq 2n$. The numerator is a perfect square: $m^2 - 4mn + 4n^2 = m^2 - 2 \cdot m \cdot 2n + (2n)^2 = (m-2n)^2$. Hence

$$\frac{m^2 - 4mn + 4n^2}{m - 2n} = \frac{(m-2n)^2}{m-2n} = m - 2n, \qquad m \neq 2n.$$

Exercise 16. Condition: $y \neq -5$. Using $25 - y^2 = (5-y)(5+y)$:

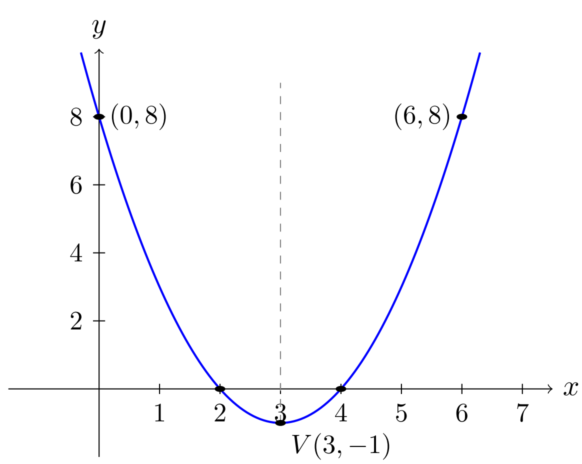
$$\frac{25 - y^2}{y+5} + y = \frac{(5-y)(5+y)}{y+5} + y = 5 - y + y = 5, \qquad y \neq -5.$$

Topic 4 – Draw the Graph of a parabola

For $f(x) = ax^2 + bx + c$ we find: the direction of opening (upwards if $a > 0$), the roots (solving $f(x) = 0$), the vertex $V = \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$, the y-intercept $(0, c)$, and its symmetric point with respect to the axis $x = -\frac{b}{2a}$.

Exercise 17. $f(x) = x^2 - 6x + 8$, with $a = 1 > 0$: the parabola opens upwards.

- Roots: $x^2 - 6x + 8 = (x-2)(x-4) = 0 \iff x = 2$ or $x = 4$.
- Vertex: $x_V = -\dfrac{-6}{2} = 3$, $f(3) = 9 - 18 + 8 = -1$, so $V = (3, -1)$. Axis of symmetry: $x = 3$.
- y-intercept: $f(0) = 8$, so $(0, 8)$; its symmetric point w.r.t. $x = 3$ is $(6, 8)$.

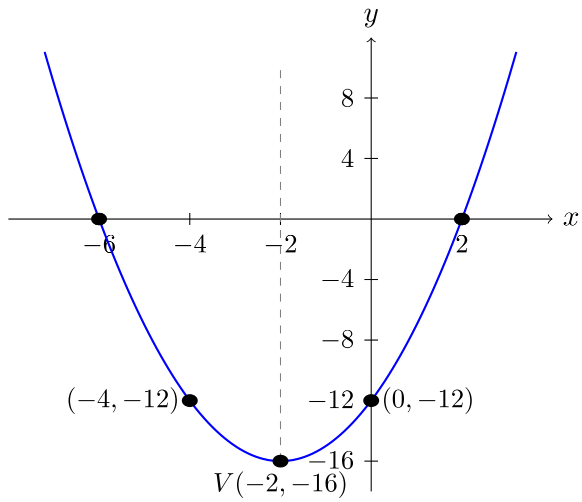


Exercise 18. First we expand:

$$f(x) = (x+2)^2 - 16 = x^2 + 4x + 4 - 16 = x^2 + 4x - 12.$$

Here $a = 1 > 0$: the parabola opens upwards.

- Roots: $(x+2)^2 = 16 \iff x+2 = \pm 4 \iff x = 2$ or $x = -6$. (Equivalently, $x^2+4x-12 = (x+6)(x-2)$.)
- Vertex: from the form $(x+2)^2 - 16$ we read directly $V = (-2, -16)$; indeed $x_V = -\frac{4}{2} = -2$ and $f(-2) = -16$. Axis of symmetry: $x = -2$.
- y-intercept: $f(0) = -12$, so $(0, -12)$; its symmetric point w.r.t. $x = -2$ is $(-4, -12)$.



Exercise 19. A parabola with roots $x = -1$ and $x = 5$ has the form

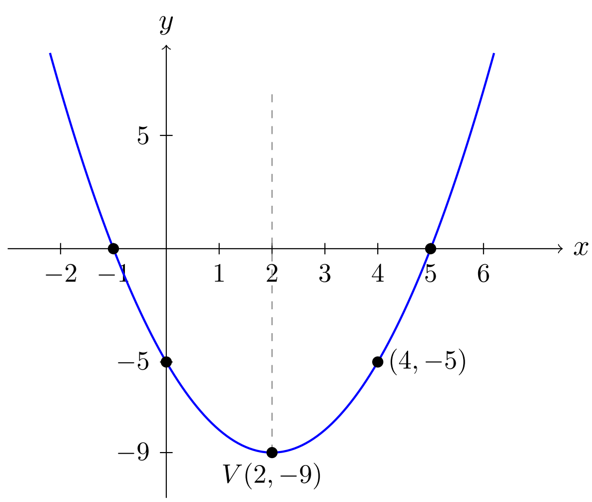
$$f(x) = a(x+1)(x-5), \qquad a \neq 0.$$

Imposing that it passes through $(0, -5)$:

$$f(0) = a(0+1)(0-5) = -5a = -5 \iff a = 1.$$

Hence $f(x) = (x+1)(x-5) = x^2 - 4x - 5$. Since $a = 1 > 0$, it opens upwards.

- Vertex: $x_V = -\dfrac{-4}{2} = 2$ (the midpoint of the roots -1 and 5), $f(2) = 4 - 8 - 5 = -9$, so $V = (2, -9)$. Axis: $x = 2$.
- y-intercept: $(0, -5)$; its symmetric point w.r.t. $x = 2$ is $(4, -5)$.

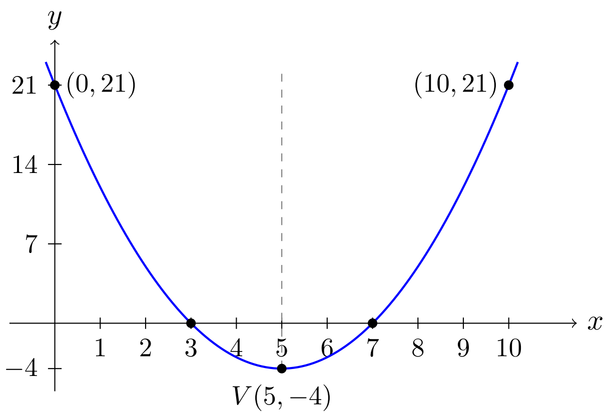


Exercise 20. Since $f(x) = x^2 + bx + c$ has leading coefficient 1 and roots 3 and 7,

$$f(x) = (x-3)(x-7) = x^2 - 10x + 21.$$

Check: $f(0) = (-3)(-7) = 21$, as required. (Equivalently: $c = f(0) = 21$, and from $f(3) = 9 + 3b + 21 = 0$ we get $b = -10$.)
The parabola opens upwards ($a = 1 > 0$).

- Vertex: $x_V = -\dfrac{-10}{2} = 5$ (the midpoint of 3 and 7), $f(5) = 25 - 50 + 21 = -4$, so $V = (5, -4)$. Axis: $x = 5$.
- y-intercept: $(0, 21)$; its symmetric point w.r.t. $x = 5$ is $(10, 21)$.



Topic 5 – Rationalizing

The key identity is $(A-B)(A+B) = A^2 - B^2$: multiplying by the *conjugate* eliminates the square roots.

Exercise 21. Multiply numerator and denominator by the conjugate $4 + \sqrt{6}$:

$$\frac{\sqrt{6}}{4-\sqrt{6}} \cdot \frac{4+\sqrt{6}}{4+\sqrt{6}} = \frac{4\sqrt{6}+6}{16-6} = \frac{4\sqrt{6}+6}{10} = \frac{2(2\sqrt{6}+3)}{10} = \frac{2\sqrt{6}+3}{5}.$$

Exercise 22. Here the square roots must be removed from the *numerator*, so we multiply by $\sqrt{7}+\sqrt{5}$:

$$\frac{\sqrt{7}-\sqrt{5}}{2} \cdot \frac{\sqrt{7}+\sqrt{5}}{\sqrt{7}+\sqrt{5}} = \frac{7-5}{2(\sqrt{7}+\sqrt{5})} = \frac{2}{2(\sqrt{7}+\sqrt{5})} = \frac{1}{\sqrt{7}+\sqrt{5}}.$$

Exercise 23. Multiply by the conjugate $\sqrt{7}+\sqrt{2}$:

$$\frac{3}{\sqrt{7}-\sqrt{2}} \cdot \frac{\sqrt{7}+\sqrt{2}}{\sqrt{7}+\sqrt{2}} = \frac{3(\sqrt{7}+\sqrt{2})}{7-2} = \frac{3(\sqrt{7}+\sqrt{2})}{5} = \frac{3\sqrt{7}+3\sqrt{2}}{5}.$$

Exercise 24. Multiply by the conjugate $2+\sqrt{3}$:

$$\frac{2+\sqrt{3}}{2-\sqrt{3}} \cdot \frac{2+\sqrt{3}}{2+\sqrt{3}} = \frac{(2+\sqrt{3})^2}{4-3} = \frac{4+4\sqrt{3}+3}{1} = 7+4\sqrt{3}.$$