Trigonometric formulas

Sum and difference, double angle, half angle

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Throughout, α, β are angles; the tangent formulas hold whenever all the tangents involved are defined and no denominator vanishes.

1 Sum and difference formulas

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta, \qquad \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta,$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta, \qquad \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta,$$
$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}, \qquad \tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}.$$

Remark. Pay attention to the signs: in the sine formulas the sign in the middle is the *same* as in the argument, in the cosine formulas it is the *opposite*. In general $\sin(\alpha+\beta) \neq \sin\alpha + \sin\beta$: for instance $\sin\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \sin\pi = 0$, while $\sin\frac{\pi}{2} + \sin\frac{\pi}{2} = 2$.

Remark (Where the tangent formula comes from). Divide numerator and denominator of $\tan(\alpha+\beta) = \dfrac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}$ by $\cos\alpha\cos\beta$.

Example 1 ($\sin 75°$)**.** Write $75° = 45° + 30°$:

$$\sin 75° = \sin 45°\cos 30° + \cos 45°\sin 30° = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}.$$

Example 2 ($\cos\frac{\pi}{12}$)**.** Write $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$ (that is $15° = 60° - 45°$):

$$\cos\frac{\pi}{12} = \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4} = \frac{1}{2}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}+\sqrt{6}}{4}.$$

Note that $\cos 15° = \sin 75°$, as expected since $15° + 75° = 90°$.

Example 3 ($\tan 105°$)**.** Write $105° = 60° + 45°$:

$$\tan 105° = \frac{\tan 60° + \tan 45°}{1 - \tan 60°\tan 45°} = \frac{\sqrt{3}+1}{1-\sqrt{3}} = \frac{(\sqrt{3}+1)(1+\sqrt{3})}{(1-\sqrt{3})(1+\sqrt{3})} = \frac{4+2\sqrt{3}}{-2} = -2-\sqrt{3}.$$

The result is negative, as it must be since $105°$ is in the II quadrant.

Example 4 (Associated angles)**.** The sum and difference formulas give back the symmetry rules of the unit circle. Using $\sin\frac{\pi}{2} = 1$, $\cos\frac{\pi}{2} = 0$, $\sin\pi = 0$, $\cos\pi = -1$:

$$\sin\left(x+\tfrac{\pi}{2}\right) = \sin x\cdot 0 + \cos x\cdot 1 = \cos x, \qquad \cos\left(x+\tfrac{\pi}{2}\right) = \cos x\cdot 0 - \sin x\cdot 1 = -\sin x,$$
$$\sin\left(\tfrac{\pi}{2}-x\right) = 1\cdot\cos x - 0\cdot\sin x = \cos x, \qquad \cos\left(\tfrac{\pi}{2}-x\right) = 0\cdot\cos x + 1\cdot\sin x = \sin x,$$
$$\sin(\pi-x) = 0\cdot\cos x - (-1)\sin x = \sin x, \qquad \cos(\pi-x) = (-1)\cos x + 0\cdot\sin x = -\cos x,$$
$$\sin(\pi+x) = -\sin x, \qquad \cos(\pi+x) = -\cos x.$$

Example 5 (Using the quadrants)**.** Let $\sin\alpha = \frac{3}{5}$ with α in the II quadrant, and $\cos\beta = \frac{5}{13}$ with β in the IV quadrant. Compute $\sin(\alpha+\beta)$, $\cos(\alpha+\beta)$, $\tan(\alpha+\beta)$.
First we find the missing values with $\sin^2+\cos^2=1$ and the signs of the quadrants:

$$\cos\alpha = -\sqrt{1-\tfrac{9}{25}} = -\frac{4}{5}, \qquad \sin\beta = -\sqrt{1-\tfrac{25}{169}} = -\frac{12}{13}.$$

Then

$$\begin{aligned}
\sin(\alpha+\beta) &= \frac{3}{5}\cdot\frac{5}{13} + \left(-\frac{4}{5}\right)\left(-\frac{12}{13}\right) = \frac{15+48}{65} = \frac{63}{65},\\
\cos(\alpha+\beta) &= \left(-\frac{4}{5}\right)\frac{5}{13} - \frac{3}{5}\left(-\frac{12}{13}\right) = \frac{-20+36}{65} = \frac{16}{65},\\
\tan(\alpha+\beta) &= \frac{63/65}{16/65} = \frac{63}{16}.
\end{aligned}$$

Check: $63^2+16^2 = 3969+256 = 4225 = 65^2$.

Example 6 (Simplifying)**.**

$$\sin\left(x+\frac{\pi}{3}\right)+\sin\left(x-\frac{\pi}{3}\right) = \left(\sin x\cdot\tfrac{1}{2}+\cos x\cdot\tfrac{\sqrt{3}}{2}\right)+\left(\sin x\cdot\tfrac{1}{2}-\cos x\cdot\tfrac{\sqrt{3}}{2}\right) = \sin x.$$

2 Double angle formulas

Taking $\beta=\alpha$ in the sum formulas:

$$\begin{aligned}
&\sin 2\alpha = 2\sin\alpha\cos\alpha,\\
&\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha,\\
&\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}.
\end{aligned}$$

The three forms of $\cos 2\alpha$ are equivalent because of $\sin^2\alpha+\cos^2\alpha=1$; use the one that contains the data you have.

Example 7 (Checking with notable angles)**.** For $\alpha=\frac{\pi}{6}$: $\sin\frac{\pi}{3} = 2\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$, $\cos\frac{\pi}{3} = 1-2\cdot\frac{1}{4} = \frac{1}{2}$, $\tan\frac{\pi}{3} = \dfrac{2\cdot\frac{\sqrt{3}}{3}}{1-\frac{1}{3}} = \dfrac{2\sqrt{3}/3}{2/3} = \sqrt{3}$.

Example 8. Let $\sin\alpha=\frac{3}{5}$ with α in the II quadrant, so $\cos\alpha=-\frac{4}{5}$. Then

$$\sin 2\alpha = 2\cdot\frac{3}{5}\cdot\left(-\frac{4}{5}\right) = -\frac{24}{25}, \qquad \cos 2\alpha = 1-2\cdot\frac{9}{25} = \frac{7}{25}, \qquad \tan 2\alpha = \frac{-24/25}{7/25} = -\frac{24}{7}.$$

Check with the formula for the tangent: $\tan\alpha = -\frac{3}{4}$ and $\dfrac{2\cdot(-3/4)}{1-9/16} = \dfrac{-3/2}{7/16} = -\dfrac{24}{7}$.

Example 9. Knowing only $\cos\alpha=\frac{1}{3}$ we can compute $\cos 2\alpha$ without knowing the quadrant:

$$\cos 2\alpha = 2\cos^2\alpha - 1 = \frac{2}{9} - 1 = -\frac{7}{9}.$$

On the other hand $\sin 2\alpha = 2\sin\alpha\cos\alpha$ requires the sign of $\sin\alpha$, that is the quadrant.

Example 10 (An identity)**.**

$$(\sin x+\cos x)^2 = \sin^2 x + 2\sin x\cos x + \cos^2 x = 1+\sin 2x.$$

Example 11 (Triple angle)**.** Combining the sum and double angle formulas:

$$\begin{aligned}\sin 3x &= \sin(2x+x) = \sin 2x\cos x + \cos 2x \sin x = 2\sin x\cos^2 x + (1-2\sin^2 x)\sin x \\ &= 2\sin x(1-\sin^2 x) + \sin x - 2\sin^3 x = 3\sin x - 4\sin^3 x.\end{aligned}$$

Check with $x = \frac{\pi}{6}$: $\sin\frac{\pi}{2} = 1$ and $3\cdot\frac{1}{2} - 4\cdot\frac{1}{8} = \frac{3}{2} - \frac{1}{2} = 1$.

3 Half angle formulas

From $\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2}$ and $\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1$ we get:

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}, \qquad \cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}},$$

$$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}, \qquad \tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha} = \frac{1-\cos\alpha}{\sin\alpha}.$$

The sign $\pm$ is chosen according to the quadrant of $\dfrac{\alpha}{2}$ (not of α!). The last two formulas for the tangent have no ambiguity of sign.

Example 12 ($\sin 15°$)**.** $15° = \frac{30°}{2}$ is in the I quadrant, so we take the + sign:

$$\sin 15° = \sqrt{\frac{1-\cos 30°}{2}} = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}} = \frac{\sqrt{2-\sqrt{3}}}{2}.$$

(With the difference formula one finds $\sin 15° = \frac{\sqrt{6}-\sqrt{2}}{4}$: the two results are equal, as one can check by squaring.)

Example 13 ($\cos\frac{\pi}{8}$ and $\tan\frac{\pi}{8}$)**.** $\frac{\pi}{8} = \frac{1}{2}\cdot\frac{\pi}{4}$ is in the I quadrant:

$$\cos\frac{\pi}{8} = \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} = \frac{\sqrt{2+\sqrt{2}}}{2},$$

$$\tan\frac{\pi}{8} = \frac{\sin\frac{\pi}{4}}{1+\cos\frac{\pi}{4}} = \frac{\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}} = \frac{\sqrt{2}}{2+\sqrt{2}} = \frac{\sqrt{2}(2-\sqrt{2})}{4-2} = \sqrt{2}-1.$$

Example 14 (Choosing the sign)**.** Let $\cos\alpha = \frac{1}{9}$ with $\frac{3\pi}{2} < \alpha < 2\pi$ (IV quadrant). Then $\frac{3\pi}{4} < \frac{\alpha}{2} < \pi$, so $\frac{\alpha}{2}$ is in the II quadrant: $\sin\frac{\alpha}{2} > 0$, $\cos\frac{\alpha}{2} < 0$.

$$\sin\frac{\alpha}{2} = \sqrt{\frac{1-\frac{1}{9}}{2}} = \sqrt{\frac{4}{9}} = \frac{2}{3}, \qquad \cos\frac{\alpha}{2} = -\sqrt{\frac{1+\frac{1}{9}}{2}} = -\sqrt{\frac{5}{9}} = -\frac{\sqrt{5}}{3},$$

$$\tan\frac{\alpha}{2} = \frac{2/3}{-\sqrt{5}/3} = -\frac{2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}.$$

Check with $\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha}$: in the IV quadrant $\sin\alpha = -\sqrt{1-\frac{1}{81}} = -\frac{4\sqrt{5}}{9}$, so $\dfrac{-4\sqrt{5}/9}{10/9} = -\dfrac{2\sqrt{5}}{5}$.

Example 15 (Squares of sine and cosine)**.** The half angle formulas are often used in the form

$$\sin^2 x = \frac{1-\cos 2x}{2}, \qquad \cos^2 x = \frac{1+\cos 2x}{2},$$

which turn a square into a first power. For instance, for $x = \frac{\pi}{12}$: $\sin^2\frac{\pi}{12} = \dfrac{1-\cos\frac{\pi}{6}}{2} = \dfrac{2-\sqrt{3}}{4}$.