Exercises: Exact Values of Trigonometric Functions

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Tools

Recall that $2\pi = 360^\circ$, so $1^\circ = \dfrac{\pi}{180}$ rad.

Known values.

α	30°	45°	60°
$\sin\alpha$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\alpha$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$

Addition formulas.

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta, \qquad \cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta.$$

Half-angle formulas. From $\cos(2\theta) = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$ we get

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}, \qquad \sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}},$$

where the sign is decided by the quadrant in which $\alpha/2$ lies.

Periodicity and parity. For every α and every $k \in \mathbb{Z}$:

$$\cos(\alpha + k\cdot 360^\circ) = \cos\alpha, \qquad \sin(\alpha + k\cdot 360^\circ) = \sin\alpha,$$

$$\cos(-\alpha) = \cos\alpha, \qquad \sin(-\alpha) = -\sin\alpha.$$

Exercise 1: $\sin(75^\circ)$

Solution

We write 75° as a sum of angles with known values: $75^\circ = 45^\circ + 30^\circ$. By the addition formula for the sine,

$$\sin(75^\circ) = \sin 45^\circ\cos 30^\circ + \cos 45^\circ\sin 30^\circ = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}.$$

Hence

$$\boxed{\sin(75^\circ) = \frac{\sqrt{6}+\sqrt{2}}{4} \approx 0.9659.}$$

Exercise 2: $\cos(105^\circ)$

Solution

We write $105^\circ = 60^\circ + 45^\circ$. By the addition formula for the cosine,

$$\cos(105^\circ) = \cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ = \frac{1}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}.$$

Hence

$$\boxed{\cos(105^\circ) = \frac{\sqrt{2}-\sqrt{6}}{4} \approx -0.2588.}$$

Sign check. 105° lies in the second quadrant ($90^\circ < 105^\circ < 180^\circ$), where the cosine is negative; indeed $\sqrt{2} < \sqrt{6}$, so the result is negative.

Exercise 3: $\cos(715^\circ)$

Solution

Step 1: reduce the angle using periodicity. The angle 715° is larger than a full turn, so we first remove as many full turns (360°) as possible. Dividing 715 by 360:

$$715 = 1\cdot 360 + 355, \qquad 0 \le 355 < 360.$$

Hence $715^\circ = 355^\circ + 360^\circ$ and, by periodicity,

$$\cos(715^\circ) = \cos(355^\circ).$$

Equivalently, in radians: $715^\circ = 715\cdot\dfrac{\pi}{180} = \dfrac{143\pi}{36} = 2\pi + \dfrac{71\pi}{36}$.

Step 2: locate the angle. Since $270^\circ < 355^\circ < 360^\circ$, the angle lies in the fourth quadrant, where the cosine is *positive*. It is 5° short of a full turn:

$$355^\circ = 360^\circ - 5^\circ.$$

Step 3: reduce to the first quadrant. Using periodicity once more and then the parity of the cosine,

$$\cos(355^\circ) = \cos(360^\circ - 5^\circ) = \cos(-5^\circ) = \cos(5^\circ).$$

(Alternatively, by the subtraction formula: $\cos(360^\circ - 5^\circ) = \cos 360^\circ \cos 5^\circ + \sin 360^\circ \sin 5^\circ = 1\cdot\cos 5^\circ + 0\cdot\sin 5^\circ = \cos 5^\circ$.)

Summing up, in one line:

$$\cos(715^\circ) = \cos(2\cdot 360^\circ - 5^\circ) = \cos(-5^\circ) = \cos(5^\circ).$$

Step 4: what can be said about $\cos(5^\circ)$. Unlike the other angles in this sheet, 5° cannot be obtained from 30°, 45°, 60° by sums, differences and halvings (all such combinations are multiples of $\frac{15^\circ}{2^n}$; in particular $5^\circ = \frac{15^\circ}{3}$ would require *dividing an angle by three*).

What we can do is relate $\cos 5^\circ$ to $\cos 15^\circ$ through the triple-angle formula. From the addition formulas,

$$\cos(3\theta) = \cos(2\theta+\theta) = \cos 2\theta\cos\theta - \sin 2\theta\sin\theta = (2\cos^2\theta - 1)\cos\theta - 2\sin^2\theta\cos\theta,$$

and using $\sin^2\theta = 1 - \cos^2\theta$,

$$\cos(3\theta) = 4\cos^3\theta - 3\cos\theta.$$

With $\theta = 5^\circ$ and $c = \cos 5^\circ$, and $\cos 15^\circ = \frac{\sqrt{6}+\sqrt{2}}{4}$ (Exercise 4), c is a root of the cubic equation

$$4c^3 - 3c - \frac{\sqrt{6}+\sqrt{2}}{4} = 0.$$

This cubic has three real roots ($\cos 5^\circ$, $\cos 125^\circ$, $\cos 245^\circ$), which is the so-called *casus irreducibilis*: Cardano's formula expresses them only through cube roots of non-real complex numbers, and there is no expression of $\cos 5^\circ$ using real square and cube roots alone. Hence the final answer is the reduced form

$$\boxed{\cos(715^\circ) = \cos(5^\circ) \approx 0.9962,}$$

and the content of the exercise is the reduction of the angle. The Appendix shows how the cubic can nevertheless be solved explicitly with complex numbers.

Exercise 4: $\cos(15^\circ)$, not using the formulas above

Solution

We do not use the addition formulas; instead we use the half-angle formula, since $15^\circ = \dfrac{30^\circ}{2}$:

$$\cos(15^\circ) = \pm\sqrt{\frac{1+\cos 30^\circ}{2}}.$$

Sign. 15° lies in the first quadrant, where the cosine is positive, so we take the + sign.

Computation.

$$\cos(15^\circ) = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{2+\sqrt{3}}{4}} = \frac{\sqrt{2+\sqrt{3}}}{2}.$$

Removing the nested radical. We look for $a, b > 0$ with $\sqrt{2+\sqrt{3}} = \sqrt{a} + \sqrt{b}$. Squaring,

$$2+\sqrt{3} = a + b + 2\sqrt{ab} \quad \Longleftarrow \quad a+b=2, \qquad 2\sqrt{ab} = \sqrt{3} \iff ab = \tfrac{3}{4}.$$

So a, b are the roots of $t^2 - 2t + \frac{3}{4} = 0$, i.e. $t = \frac{3}{2}$ and $t = \frac{1}{2}$. Therefore

$$\sqrt{2+\sqrt{3}} = \sqrt{\tfrac{3}{2}} + \sqrt{\tfrac{1}{2}} = \frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2} = \frac{\sqrt{6}+\sqrt{2}}{2}.$$

Check: $\left(\dfrac{\sqrt{6}+\sqrt{2}}{2}\right)^2 = \dfrac{6+2\sqrt{12}+2}{4} = \dfrac{8+4\sqrt{3}}{4} = 2+\sqrt{3}.$

Hence

$$\boxed{\cos(15^\circ) = \frac{\sqrt{2+\sqrt{3}}}{2} = \frac{\sqrt{6}+\sqrt{2}}{4} \approx 0.9659.}$$

Consistency check. Since $\cos(15^\circ) = \sin(90^\circ - 15^\circ) = \sin(75^\circ)$, the result must coincide with Exercise 1, and it does.

Exercise 5: $\cos(7.5^\circ)$

Solution

Since $7.5^\circ = \dfrac{15^\circ}{2}$, we apply the half-angle formula once more, using the value of $\cos 15^\circ$ from Exercise 4. The angle 7.5° is in the first quadrant, so the sign is $+$:

$$\cos(7.5^\circ) = \sqrt{\frac{1+\cos 15^\circ}{2}}.$$

First form. Using $\cos 15^\circ = \dfrac{\sqrt{2+\sqrt{3}}}{2}$:

$$\cos(7.5^\circ) = \sqrt{\frac{1+\frac{\sqrt{2+\sqrt{3}}}{2}}{2}} = \sqrt{\frac{2+\sqrt{2+\sqrt{3}}}{4}} = \frac{\sqrt{2+\sqrt{2+\sqrt{3}}}}{2}.$$

Second form. Using $\cos 15^\circ = \dfrac{\sqrt{6}+\sqrt{2}}{4}$:

$$\cos(7.5^\circ) = \sqrt{\frac{1+\frac{\sqrt{6}+\sqrt{2}}{4}}{2}} = \sqrt{\frac{4+\sqrt{6}+\sqrt{2}}{8}} = \sqrt{\frac{8+2\sqrt{6}+2\sqrt{2}}{16}} = \frac{\sqrt{8+2\sqrt{6}+2\sqrt{2}}}{4}.$$

Hence

$$\boxed{\cos(7.5^\circ) = \frac{\sqrt{2+\sqrt{2+\sqrt{3}}}}{2} = \frac{\sqrt{8+2\sqrt{6}+2\sqrt{2}}}{4} \approx 0.9914.}$$

Remark. The pattern is general: each halving of the angle adds one nested "$\sqrt{2+\cdots}$", since $\cos\frac{\alpha}{2} = \frac{1}{2}\sqrt{2+2\cos\alpha}$ for $0 \le \alpha \le 180^\circ$. Starting from $2\cos 30^\circ = \sqrt{3}$:

$$2\cos 30^\circ = \sqrt{3}, \qquad 2\cos 15^\circ = \sqrt{2+\sqrt{3}}, \qquad 2\cos 7.5^\circ = \sqrt{2+\sqrt{2+\sqrt{3}}}, \ \ldots$$

Appendix: computing $\cos(5^\circ)$ with the cubic equation

Throughout, we write $k = \cos 15^\circ = \dfrac{\sqrt{6}+\sqrt{2}}{4}$ and $s = \sin 15^\circ$. From Exercise 3 we know that $c = \cos 5^\circ$ satisfies

$$4c^3 - 3c - k = 0. \tag{$*$}$$

A.1 Complex numbers we need

Every nonzero complex number can be written in *trigonometric form*

$$z = \rho\,(\cos\varphi + i\sin\varphi), \qquad \rho = |z| > 0,$$

and multiplication multiplies the moduli and adds the arguments. In particular we have *De Moivre's formula*

$$\big(\cos\varphi + i\sin\varphi\big)^n = \cos(n\varphi) + i\sin(n\varphi).$$

Consequently, the equation $w^3 = z$ has exactly three solutions, the *cube roots* of z:

$$w_j = \sqrt[3]{\rho}\left(\cos\tfrac{\varphi+360^\circ j}{3} + i\sin\tfrac{\varphi+360^\circ j}{3}\right), \qquad j = 0, 1, 2.$$

Their arguments are $\frac{\varphi}{3}$, $\frac{\varphi}{3}+120^\circ$, $\frac{\varphi}{3}+240^\circ$: the three cube roots are the vertices of an equilateral triangle centred at the origin.

We also need the conjugate $\bar z = \rho(\cos\varphi - i\sin\varphi)$, and the facts $z\bar z = \rho^2$, $z + \bar z = 2\rho\cos\varphi$.

A.2 Reducing to a depressed cubic

Any cubic equation $ax^3 + bx^2 + cx + d = 0$ can be brought, by dividing by a and substituting $x = t - \frac{b}{3a}$, to the *depressed* form

$$t^3 + pt + q = 0,$$

without a term in t^2. Our equation $(*)$ already has no c^2 term; dividing by 4 we get

$$c^3 - \frac{3}{4}\,c - \frac{k}{4} = 0, \qquad \text{i.e.} \qquad p = -\frac{3}{4}, \quad q = -\frac{k}{4}.$$

A.3 Cardano's method

The idea is to look for a solution of $t^3 + pt + q = 0$ of the form $t = u + v$. Expanding,

$$(u+v)^3 = u^3 + v^3 + 3uv(u+v),$$

so the equation becomes

$$u^3 + v^3 + (3uv + p)(u+v) + q = 0.$$

We have one equation and two unknowns, so we are free to impose one extra condition. We choose

$$3uv + p = 0, \qquad \text{i.e.} \qquad uv = -\frac{p}{3},$$

which kills the middle term. We are left with the system

$$u^3 + v^3 = -q, \qquad u^3 v^3 = -\frac{p^3}{27}.$$

So u^3 and v^3 are two numbers whose sum and product are known: they are the roots of the *quadratic resolvent*

$$z^2 + qz - \frac{p^3}{27} = 0.$$

In this way a cubic equation is reduced to a quadratic equation followed by the extraction of cube roots.

A.4 Applying it to our equation

With $p = -\frac{3}{4}$, $q = -\frac{k}{4}$ the conditions become

$$uv = \frac{1}{4}, \qquad u^3 + v^3 = \frac{k}{4}, \qquad u^3 v^3 = \frac{1}{64},$$

and the resolvent is

$$z^2 - \frac{k}{4}\, z + \frac{1}{64} = 0.$$

Its discriminant is

$$\Delta = \frac{k^2}{16} - \frac{4}{64} = \frac{k^2 - 1}{16} = -\frac{\sin^2 15^\circ}{16} < 0.$$

So the resolvent has *no real roots*: its roots are the complex conjugates

$$z_{1,2} = \frac{\frac{k}{4} \pm i\,\frac{s}{2}}{2} = \frac{k \pm is}{8} = \frac{1}{8}\big(\cos 15^\circ \pm i \sin 15^\circ\big).$$

This is remarkable: we are looking for a *real* number, $\cos 5^\circ$, and the method forces us through non-real complex numbers.

Cube roots. We need $u^3 = z_1 = \frac{1}{8}(\cos 15^\circ + i \sin 15^\circ)$. Here $\rho = \frac{1}{8}$, $\sqrt[3]{\rho} = \frac{1}{2}$ and $\varphi = 15^\circ$, so by A.1

$$u_j = \frac{1}{2}\big(\cos(5^\circ + 120^\circ j) + i \sin(5^\circ + 120^\circ j)\big), \qquad j = 0, 1, 2.$$

Pairing u and v. The value of v is not chosen independently: the condition $uv = \frac{1}{4}$ forces $v = \dfrac{1}{4u}$. Since $|u_j| = \frac{1}{2}$, we get $\frac{1}{4u_j} = \bar{u}_j$, so

$$v_j = \bar{u}_j = \frac{1}{2}\big(\cos(5^\circ + 120^\circ j) - i \sin(5^\circ + 120^\circ j)\big),$$

which is indeed a cube root of $z_2 = \bar{z}_1$.

The three roots. Finally $c_j = u_j + v_j = u_j + \bar{u}_j = 2 \operatorname{Re} u_j$, so

$$c_j = \cos(5^\circ + 120^\circ j), \qquad j = 0, 1, 2,$$

that is

$$c_0 = \cos 5^\circ \approx 0.9962, \qquad c_1 = \cos 125^\circ \approx -0.5736, \qquad c_2 = \cos 245^\circ \approx -0.4226.$$

The imaginary parts cancel and all three roots are real, as expected. The one we want is the only root in $(0, 1)$, namely c_0.

A.5 The closed formula

Since $\sin 15^\circ = \sqrt{\frac{1 - \cos 30^\circ}{2}} = \dfrac{\sqrt{2 - \sqrt{3}}}{2} = \dfrac{\sqrt{6} - \sqrt{2}}{4}$ (half-angle formula, denested as in Exercise 4), Cardano's formula $c = u + v = \sqrt[3]{z_1} + \sqrt[3]{z_2}$ gives

$$\boxed{\cos 5^\circ = \frac{1}{2}\left(\sqrt[3]{\frac{\sqrt{6} + \sqrt{2}}{4} + i\,\frac{\sqrt{6} - \sqrt{2}}{4}} + \sqrt[3]{\frac{\sqrt{6} + \sqrt{2}}{4} - i\,\frac{\sqrt{6} - \sqrt{2}}{4}}\right),}$$

where the cube roots are the principal ones (with arguments 5° and -5° respectively). This is an exact expression of $\cos 5^\circ$ using only rational numbers, $+, -, \times, \div$, square roots, cube roots and i. A numerical check gives $0.9961946\ldots$, in agreement with $\cos 5^\circ$.

A.6 Why the complex numbers cannot be avoided

For a depressed cubic with real coefficients, the sign of

$$D = -\bigl(4p^3 + 27q^2\bigr)$$

decides the nature of the roots: $D > 0$ means three distinct real roots, $D < 0$ one real root and two complex conjugate ones. Here

$$D = -\Bigl(4 \cdot \Bigl(-\frac{27}{64}\Bigr) + 27 \cdot \frac{k^2}{16}\Bigr) = \frac{27}{16}\bigl(1 - k^2\bigr) = \frac{27}{16}\sin^2 15^\circ > 0,$$

confirming that there are three real roots. But $D > 0$ is exactly the case in which the resolvent of A.3 has negative discriminant (indeed $\Delta = q^2 + \frac{4p^3}{27} = -\frac{D}{27}$), so Cardano's formula *always* goes through non-real numbers when all three roots are real. This is the classical *casus irreducibilis*. A theorem of algebra (whose proof uses Galois theory and is beyond these notes) states that, when such a cubic is irreducible, as it is here, its roots cannot be written using real radicals only. So the complex expression of A.5 is the best possible: $\cos 5^\circ$ has no formula with nested real square and cube roots like those found for 15° and 7.5°.

A remark on circularity. Note that to compute the principal cube root of $\cos 15^\circ + i \sin 15^\circ$ in A.4 we used De Moivre's formula, i.e. we divided the angle 15° by three, which amounts to knowing $\cos 5^\circ$ already. The algebraic formula of A.5 is exact, but evaluating it numerically is no easier than evaluating $\cos 5^\circ$ directly. Solving the cubic and trisecting the angle are, in a precise sense, the same problem.