Exercises I–V: Injectivity, Inverse Functions, and Restricting Domain and Codomain

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Throughout, "one-to-one" is used on the board in the sense of "invertible": a function $f\colon A \to B$ has an inverse $f^{-1}\colon B \to A$ if and only if it is *bijective*, i.e. both

- injective: $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$;
- surjective: for every $y \in B$ there exists $x \in A$ with $f(x) = y$.

In practice both properties are checked at once by solving the equation $f(x) = y$ for a generic $y \in B$: f is bijective exactly when, for every $y \in B$, this equation has *exactly one* solution $x \in A$, and that solution is $f^{-1}(y)$.

Exercise I

Exercise. *Consider*

$$F\colon \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+, \qquad x \longmapsto x^2,$$

where $\mathbb{R}_0^+ = [0, +\infty)$. *Find the inverse of* F.

Solution

Injectivity. Let $x_1, x_2 \geq 0$ with $x_1^2 = x_2^2$. Then

$$0 = x_1^2 - x_2^2 = (x_1 - x_2)(x_1 + x_2).$$

If $x_1 + x_2 = 0$, since both are ≥ 0 we get $x_1 = x_2 = 0$. Otherwise $x_1 + x_2 > 0$ and we can divide, obtaining $x_1 = x_2$. In both cases $x_1 = x_2$, so F is injective.

Note that the restriction of the domain to $\mathbb{R}_0^+$ is essential: on $\mathbb{R}$ the function $x \mapsto x^2$ is not injective, since e.g. $(-1)^2 = 1^2$.

Surjectivity. Let $y \geq 0$. The equation $x^2 = y$ has, in $\mathbb{R}$, the two solutions $x = \pm\sqrt{y}$ (which coincide when $y = 0$); exactly one of them, $x = \sqrt{y}$, lies in the domain $\mathbb{R}_0^+$. Hence every $y \geq 0$ is attained.

Again the restriction of the codomain matters: with codomain $\mathbb{R}$ the function would not be surjective, since $x^2 = -1$ has no real solution.

Inverse. F is bijective, and from the computation above

$$F^{-1}\colon \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+, \qquad y \longmapsto \sqrt{y}.$$

Check. For $x \geq 0$: $F^{-1}(F(x)) = \sqrt{x^2} = |x| = x$ (here we use $x \geq 0$).
For $y \geq 0$: $F(F^{-1}(y)) = (\sqrt{y})^2 = y$.

Exercise II

Exercise. *Consider*

$$G\colon \mathbb{R} \longrightarrow \mathbb{R}, \qquad x \longmapsto \frac{3x-1}{2}.$$

Is G one-to-one? If yes, find its inverse.

Solution

Injectivity. If $G(x_1) = G(x_2)$ then

$$\frac{3x_1-1}{2} = \frac{3x_2-1}{2} \implies 3x_1 - 1 = 3x_2 - 1 \implies 3x_1 = 3x_2 \implies x_1 = x_2.$$

So G is injective.

Surjectivity. Let $y \in \mathbb{R}$. We solve $G(x) = y$:

$$\frac{3x-1}{2} = y \iff 3x - 1 = 2y \iff x = \frac{2y+1}{3}.$$

For every $y \in \mathbb{R}$ this gives exactly one $x \in \mathbb{R}$, so G is surjective.

Answer. Yes, G is bijective, with inverse

$$G^{-1}\colon \mathbb{R} \longrightarrow \mathbb{R}, \qquad y \longmapsto \frac{2y+1}{3}.$$

Check. For $x \in \mathbb{R}$: $G^{-1}(G(x)) = \dfrac{2 \cdot \frac{3x-1}{2} + 1}{3} = \dfrac{3x-1+1}{3} = x.$

For $y \in \mathbb{R}$: $G(G^{-1}(y)) = \dfrac{3 \cdot \frac{2y+1}{3} - 1}{2} = \dfrac{2y+1-1}{2} = y.$

Remark. More generally, every affine function $x \mapsto ax + b$ with $a \neq 0$ is a bijection $\mathbb{R} \to \mathbb{R}$ with inverse $y \mapsto (y-b)/a$; here $a = \frac{3}{2}$, $b = -\frac{1}{2}$.

Exercise III

Exercise. *Consider*

$$H\colon \mathbb{R} \setminus \{-3\} \longrightarrow \mathbb{R}, \qquad x \longmapsto \frac{3x-1}{x+3}.$$

Is H one-to-one? If yes, find its inverse.

Solution

Domain. The point -3 is excluded because it makes the denominator vanish; on $\mathbb{R} \setminus \{-3\}$ the expression is well defined.

Injectivity. Let $x_1, x_2 \neq -3$ with $H(x_1) = H(x_2)$. Multiplying by the (nonzero) denominators:

$$(3x_1 - 1)(x_2 + 3) = (3x_2 - 1)(x_1 + 3).$$

Expanding both sides:

$$3x_1x_2 + 9x_1 - x_2 - 3 = 3x_1x_2 + 9x_2 - x_1 - 3,$$

that is $10x_1 = 10x_2$, so $x_1 = x_2$. Hence H is injective.

Surjectivity. Let $y \in \mathbb{R}$ and solve $H(x) = y$ with $x \neq -3$:

$$\frac{3x-1}{x+3} = y \iff 3x - 1 = y(x+3) \iff 3x - yx = 3y + 1 \iff (3-y)\,x = 3y + 1.$$

- If $y \neq 3$, we get the unique solution $x = \dfrac{3y+1}{3-y}$. This x is admissible, i.e. $x \neq -3$: indeed $\dfrac{3y+1}{3-y} = -3$ would give $3y + 1 = -9 + 3y$, i.e. $1 = -9$, which is false.
- If $y = 3$, the equation becomes $0 \cdot x = 10$, which has no solution.

Therefore the value 3 is *not* attained, and the image of H is

$$H\big(\mathbb{R} \setminus \{-3\}\big) = \mathbb{R} \setminus \{3\}.$$

(Intuitively, $y = 3$ is the horizontal asymptote of H: indeed $H(x) = 3 - \dfrac{10}{x+3}$, and $\dfrac{10}{x+3}$ never vanishes.)

Answer. H is injective, but with codomain $\mathbb{R}$ it is *not* surjective, hence not invertible as stated. Replacing the codomain with its image, the function

$$H\colon \mathbb{R} \setminus \{-3\} \longrightarrow \mathbb{R} \setminus \{3\}, \qquad x \longmapsto \frac{3x-1}{x+3}$$

is bijective, with inverse

$$H^{-1}\colon \mathbb{R} \setminus \{3\} \longrightarrow \mathbb{R} \setminus \{-3\}, \qquad y \longmapsto \frac{3y+1}{3-y}.$$

Check. For $x \neq -3$, put $y = H(x) = \dfrac{3x-1}{x+3}$. Then

$$3y + 1 = \frac{3(3x-1) + (x+3)}{x+3} = \frac{10x}{x+3}, \qquad 3 - y = \frac{3(x+3) - (3x-1)}{x+3} = \frac{10}{x+3},$$

so $H^{-1}(H(x)) = \dfrac{10x}{10} = x$.

For $y \neq 3$, put $x = H^{-1}(y) = \dfrac{3y+1}{3-y}$. Then

$$3x - 1 = \frac{3(3y+1) - (3-y)}{3-y} = \frac{10y}{3-y}, \qquad x + 3 = \frac{(3y+1) + 3(3-y)}{3-y} = \frac{10}{3-y},$$

so $H(H^{-1}(y)) = \dfrac{10y}{10} = y$.

Exercise IV

Exercise. *Consider*

$$K\colon \mathbb{R} \longrightarrow \mathbb{R}, \qquad x \longmapsto 3^x - 2.$$

Change the domain and the codomain of K in such a way as to make K one-to-one (bijective), and find its inverse.

Solution

Step 1: the domain. The expression $3^x - 2$ is defined for every $x \in \mathbb{R}$, so there is no need to shrink the domain: we keep $\mathbb{R}$.

Step 2: injectivity. The function $x \mapsto 3^x$ is strictly increasing on $\mathbb{R}$ (its base is $3 > 1$), and subtracting the constant 2 does not change this. Hence K is strictly increasing, and therefore injective: if $x_1 < x_2$ then $3^{x_1} - 2 < 3^{x_2} - 2$, so $x_1 \neq x_2 \Rightarrow K(x_1) \neq K(x_2)$.

Step 3: the image, i.e. the new codomain. Since 3^x takes exactly all values in $(0, +\infty)$ as x ranges over $\mathbb{R}$,

$$K(\mathbb{R}) = \{\, 3^x - 2 : x \in \mathbb{R} \,\} = (0 - 2,\, +\infty) = (-2, +\infty).$$

With the codomain $\mathbb{R}$ the function is *not* surjective (e.g. the value -3 is never attained, because $3^x - 2 = -3$ would require $3^x = -1$). We therefore replace the codomain with the image $(-2, +\infty)$.

Explicitly, surjectivity onto $(-2, +\infty)$: given $y > -2$, we look for x with $3^x - 2 = y$, i.e. $3^x = y + 2$. Since $y + 2 > 0$, the solution exists and is $x = \log_3(y + 2)$.

Conclusion. The function

$$K \colon \mathbb{R} \longrightarrow (-2, +\infty), \qquad x \longmapsto 3^x - 2$$

is bijective.

Step 4: the inverse. Solving $y = 3^x - 2$ for x:

$$y = 3^x - 2 \iff y + 2 = 3^x \iff x = \log_3(y + 2).$$

Hence

$$K^{-1} \colon (-2, +\infty) \longrightarrow \mathbb{R}, \qquad y \longmapsto \log_3(y + 2).$$

Check. For $x \in \mathbb{R}$: $K^{-1}(K(x)) = \log_3\big((3^x - 2) + 2\big) = \log_3 3^x = x$.
For $y > -2$: $K(K^{-1}(y)) = 3^{\log_3(y+2)} - 2 = (y + 2) - 2 = y$.

Exercise V

Exercise. *Consider*

$$T \colon \mathbb{R} \longrightarrow \mathbb{R}, \qquad x \longmapsto \log_{10}(2^x - 1)\,.$$

Same as above: change the domain and the codomain of T in such a way as to make T one-to-one (bijective), and find its inverse.

Solution

Step 1: the domain. Here $\mathbb{R}$ is not an admissible domain: the logarithm requires a strictly positive argument, so we need

$$2^x - 1 > 0 \iff 2^x > 1 = 2^0 \iff x > 0,$$

where the last equivalence holds because $t \mapsto 2^t$ is strictly increasing. For instance $T(0) = \log_{10} 0$ and $T(-1) = \log_{10}(-\frac{1}{2})$ are not defined. The (largest possible) domain is therefore $(0, +\infty)$.

Step 2: injectivity. On $(0, +\infty)$, T is the composition of

$$x \longmapsto 2^x - 1 \quad \text{(strictly increasing, with values in } (0, +\infty))$$

and

$$u \longmapsto \log_{10} u \quad \text{(strictly increasing on } (0, +\infty)).$$

A composition of strictly increasing functions is strictly increasing, so T is strictly increasing on $(0, +\infty)$, hence injective.

Step 3: the image, i.e. the new codomain. As x ranges over $(0,+\infty)$, 2^x ranges over $(1,+\infty)$, so $u = 2^x - 1$ ranges over all of $(0,+\infty)$. The logarithm $\log_{10}$ maps $(0,+\infty)$ onto $\mathbb{R}$. Hence

$$T\big((0,+\infty)\big) = \mathbb{R},$$

and the codomain $\mathbb{R}$ can be kept.

Explicitly, surjectivity: given $y \in \mathbb{R}$, we look for $x > 0$ with $\log_{10}(2^x - 1) = y$, i.e. $2^x - 1 = 10^y$, i.e. $2^x = 1 + 10^y$. Since $1 + 10^y > 1$, the solution is $x = \log_2(1 + 10^y)$, and it satisfies $x > 0$, so it lies in the domain.

Conclusion. The function

$$T\colon (0,+\infty) \longrightarrow \mathbb{R}, \qquad x \longmapsto \log_{10}(2^x - 1)$$

is bijective.

Step 4: the inverse. Solving $y = \log_{10}(2^x - 1)$ for x:

$$y = \log_{10}(2^x - 1) \iff 10^y = 2^x - 1 \iff 2^x = 1 + 10^y \iff x = \log_2(1 + 10^y)\,.$$

Hence

$$T^{-1}\colon \mathbb{R} \longrightarrow (0,+\infty), \qquad y \longmapsto \log_2(1 + 10^y)\,.$$

Check. For $x > 0$:

$$T^{-1}(T(x)) = \log_2\Big(1 + 10^{\log_{10}(2^x - 1)}\Big) = \log_2\big(1 + 2^x - 1\big) = \log_2 2^x = x.$$

For $y \in \mathbb{R}$:

$$T(T^{-1}(y)) = \log_{10}\Big(2^{\log_2(1+10^y)} - 1\Big) = \log_{10}\big(1 + 10^y - 1\big) = \log_{10} 10^y = y.$$

Remark. The two exercises illustrate the two possible fixes. In Exercise IV the domain was fine but the codomain was too large (surjectivity failed), so we shrank the codomain to the image. In Exercise V the function was not even defined on all of $\mathbb{R}$, so we shrank the domain to the natural domain; the codomain $\mathbb{R}$ was then already the image.