

Exponential and logarithmic equations – Injective, surjective and
inverse functions
Exercises with solutions

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Part A – Equations

Solve in $\mathbb{R}$, stating the domain conditions.

A1. $\log_2(x+1) + \log_2(x-1) = 3 + \log_2(x-2)$

Solution.

Domain. All three arguments must be positive:

$$\begin{cases} x+1 > 0 \\ x-1 > 0 \\ x-2 > 0 \end{cases} \implies \begin{cases} x > -1 \\ x > 1 \\ x > 2 \end{cases} \implies x > 2.$$

Solving. For $x > 2$, on the left $\log_2(x+1) + \log_2(x-1) = \log_2[(x+1)(x-1)] = \log_2(x^2-1)$. On the right, write $3 = \log_2 8$: $3 + \log_2(x-2) = \log_2 8 + \log_2(x-2) = \log_2[8(x-2)]$. The equation becomes

$$\log_2(x^2-1) = \log_2[8(x-2)].$$

Since $\log_2$ is injective, the arguments are equal:

$$x^2-1 = 8x-16 \implies x^2-8x+15 = 0 \implies (x-3)(x-5) = 0.$$

Both $x=3$ and $x=5$ satisfy $x > 2$, so both are acceptable.

Check. $x=3$: $\log_2 4 + \log_2 2 = 2 + 1 = 3$ and $3 + \log_2 1 = 3$. ✓

$x=5$: $\log_2 6 + \log_2 4 = \log_2 24$ and $3 + \log_2 3 = \log_2 8 + \log_2 3 = \log_2 24$. ✓

$x=3, \quad x=5$

A2. $\log_2(4^x-3) = x+1$

Solution.

Domain. The argument must be positive: $4^x-3 > 0$, i.e. $4^x > 3$.

Solving. By definition of logarithm, $\log_2 A = B \iff A = 2^B$. Therefore

$$4^x-3 = 2^{x+1} \implies (2^x)^2-3 = 2 \cdot 2^x,$$

where we used $4^x = (2^x)^2$ and $2^{x+1} = 2 \cdot 2^x$. Set $t = 2^x$, with $t > 0$:

$$t^2-2t-3 = 0 \implies (t-3)(t+1) = 0 \implies t=3 \text{ or } t=-1.$$

The value $t=-1$ is rejected, because $t = 2^x > 0$. Hence

$$2^x = 3 \implies x = \log_2 3.$$

Domain check. $4^x = (2^x)^2 = 3^2 = 9 > 3$. ✓

Check. $\log_2(9-3) = \log_2 6 = \log_2(2 \cdot 3) = 1 + \log_2 3 = x+1$. ✓

$x = \log_2 3$

Part B – Functions

The exercises are ordered by increasing difficulty.

B1. Determine the natural domain (field of existence) of the following functions:

$$(a) \ f(x) = \frac{\sqrt{x+3}}{x-2}, \quad (b) \ g(x) = \log_3(x^2 - 5x + 6), \quad (c) \ h(x) = \sqrt{2^x - 8}.$$

Solution.

(a) $f(x) = \frac{\sqrt{x+3}}{x-2}$: the radicand must be non-negative and the denominator non-zero:

$$\begin{cases} x+3 \geq 0 \\ x-2 \neq 0 \end{cases} \implies \begin{cases} x \geq -3 \\ x \neq 2 \end{cases} \quad D_f = [-3, 2) \cup (2, +\infty).$$

(b) $g(x) = \log_3(x^2 - 5x + 6)$: the argument must be positive:

$$x^2 - 5x + 6 > 0 \iff (x-2)(x-3) > 0 \iff x < 2 \text{ or } x > 3, \quad D_g = (-\infty, 2) \cup (3, +\infty).$$

(The parabola $y = (x-2)(x-3)$ opens upwards, so it is positive outside the roots.)

(c) $h(x) = \sqrt{2^x - 8}$: the radicand must be non-negative:

$$2^x - 8 \geq 0 \iff 2^x \geq 2^3 \iff x \geq 3, \quad D_h = [3, +\infty),$$

where the last step holds because 2^x is strictly increasing (base $2 > 1$).

B2. Let $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = 3x - 5$. Prove, using the definitions, that f is injective and surjective. Find f^{-1} and check that $f(f^{-1}(y)) = y$.

Solution.

Injective. Assume $f(x_1) = f(x_2)$:

$$3x_1 - 5 = 3x_2 - 5 \implies 3x_1 = 3x_2 \implies x_1 = x_2.$$

Surjective. Let $y \in \mathbb{R}$ be arbitrary and solve $f(x) = y$:

$$3x - 5 = y \implies x = \frac{y+5}{3} \in \mathbb{R}.$$

A solution exists for every y , so f is surjective.

Inverse. The computation above gives $f^{-1}(y) = \frac{y+5}{3}$, $f^{-1}: \mathbb{R} \rightarrow \mathbb{R}$.

Check. $f(f^{-1}(y)) = 3 \cdot \frac{y+5}{3} - 5 = y + 5 - 5 = y$. ✓

$$f^{-1}(y) = \frac{y+5}{3}$$

B3. Let $f(x) = 2^{x-1} + 3$.

- (a) Find the domain and prove that f is injective.
- (b) Find the image of f . Is f surjective onto $\mathbb{R}$?
- (c) Find sets A, B such that $f: A \rightarrow B$ is bijective, and compute f^{-1} .

Solution.

(a) The exponential is defined for every real exponent, so $D = \mathbb{R}$. Injectivity: assume $f(x_1) = f(x_2)$:

$$2^{x_1-1} + 3 = 2^{x_2-1} + 3 \implies 2^{x_1-1} = 2^{x_2-1} \implies x_1 - 1 = x_2 - 1 \implies x_1 = x_2,$$

using the injectivity of $x \mapsto 2^x$.

(b) Since $2^{x-1} > 0$ for every x , we have $f(x) > 3$: the image is contained in $(3, +\infty)$. Conversely, let $y > 3$ and solve $f(x) = y$:

$$2^{x-1} = y - 3 \quad (\text{with } y - 3 > 0) \implies x - 1 = \log_2(y - 3) \implies x = 1 + \log_2(y - 3).$$

So every $y > 3$ is taken, and $\text{Im } f = (3, +\infty)$. f is *not* surjective onto $\mathbb{R}$: for instance $f(x) = 3$ would require $2^{x-1} = 0$, which is impossible. Graphically, the horizontal lines $y = k$ with $k \leq 3$ do not meet the graph ($y = 3$ is the horizontal asymptote).

(c) Taking $A = \mathbb{R}$ and $B = \text{Im } f = (3, +\infty)$, the function $f: \mathbb{R} \rightarrow (3, +\infty)$ is injective by (a) and surjective by (b), hence bijective. From the computation in (b):

$$f^{-1}: (3, +\infty) \rightarrow \mathbb{R}, \quad f^{-1}(y) = 1 + \log_2(y - 3).$$

Check. $f(f^{-1}(y)) = 2^{\log_2(y-3)} + 3 = (y - 3) + 3 = y$. ✓

$f^{-1}(y) = 1 + \log_2(y - 3), \quad y > 3$

B4. Let $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^2 - 4x + 3$.

- (a) Prove that f is not injective, both algebraically and with the horizontal line test.
- (b) Find the image of f . Is f surjective?
- (c) Restrict f to the largest interval of the form $[c, +\infty)$ on which it is injective, so that it becomes a bijection onto its image. Compute the inverse of the restriction.

Solution.

Completing the square: $f(x) = (x - 2)^2 - 1$. The graph is a parabola opening upwards, with vertex $V = (2, -1)$.

(a) *Algebraically:* $f(1) = 1 - 4 + 3 = 0$ and $f(3) = 9 - 12 + 3 = 0$, with $1 \neq 3$. So f is not injective.

Graphically: the line $y = 0$ meets the parabola in the two points $(1, 0)$ and $(3, 0)$. More generally, every line $y = k$ with $k > -1$ meets the parabola twice, at $x = 2 \pm \sqrt{k + 1}$.

(b) Since $(x - 2)^2 \geq 0$, we have $f(x) \geq -1$. Conversely, for $y \geq -1$:

$$(x - 2)^2 - 1 = y \implies (x - 2)^2 = y + 1 \implies x = 2 \pm \sqrt{y + 1},$$

which is real because $y + 1 \geq 0$. Hence $\text{Im } f = [-1, +\infty)$. f is not surjective onto $\mathbb{R}$: for instance $f(x) = -2$ would give $(x - 2)^2 = -1$, impossible. Graphically, the lines $y = k$ with $k < -1$ lie below the vertex and do not meet the parabola.

(c) On $[2, +\infty)$ the function is injective: if $x_1, x_2 \geq 2$ and $f(x_1) = f(x_2)$, then $(x_1 - 2)^2 = (x_2 - 2)^2$; since $x_1 - 2 \geq 0$ and $x_2 - 2 \geq 0$, taking square roots gives $x_1 - 2 = x_2 - 2$, i.e. $x_1 = x_2$. No larger interval $[c, +\infty)$ with $c < 2$ works: it would contain the two points $2 \pm \varepsilon$ (for small $\varepsilon > 0$), which have the same image.

The restriction $f: [2, +\infty) \rightarrow [-1, +\infty)$ is therefore bijective (surjectivity follows from (b), choosing the solution $2 + \sqrt{y + 1} \geq 2$). Solving $y = (x - 2)^2 - 1$ with $x \geq 2$ we must take the + sign:

$$x = 2 + \sqrt{y + 1}.$$

$f^{-1}: [-1, +\infty) \rightarrow [2, +\infty), \quad f^{-1}(y) = 2 + \sqrt{y + 1}$

B5. Let $f(x) = \frac{2x+1}{x-3}$.

- (a) Find the natural domain D of f .
- (b) Prove that f is injective on D .
- (c) Find the image of f , and explain which horizontal line does not meet the graph.
- (d) Compute f^{-1} , stating its domain.

Solution.

(a) Denominator non-zero: $x \neq 3$, so $D = \mathbb{R} \setminus \{3\}$.

(b) Let $x_1, x_2 \in D$ with $f(x_1) = f(x_2)$:

$$\frac{2x_1+1}{x_1-3} = \frac{2x_2+1}{x_2-3} \implies (2x_1+1)(x_2-3) = (2x_2+1)(x_1-3).$$

Expanding both sides:

$$2x_1x_2 - 6x_1 + x_2 - 3 = 2x_1x_2 - 6x_2 + x_1 - 3.$$

Cancelling $2x_1x_2$ and -3 on both sides and collecting terms:

$$-6x_1 - x_1 = -6x_2 - x_2 \implies -7x_1 = -7x_2 \implies x_1 = x_2.$$

So f is injective.

(c) Let $y \in \mathbb{R}$ and solve $f(x) = y$ for $x \neq 3$:

$$\frac{2x+1}{x-3} = y \implies 2x+1 = yx-3y \implies 2x-yx = -3y-1 \implies x(y-2) = 3y+1.$$

- If $y = 2$, the equation becomes $0 = 7$: no solution. So $2 \notin \text{Im } f$.
- If $y \neq 2$, then $x = \frac{3y+1}{y-2}$. This value is in D , since $\frac{3y+1}{y-2} = 3$ would give $3y+1 = 3y-6$, i.e. $1 = -6$, impossible.

Hence $\text{Im } f = \mathbb{R} \setminus \{2\}$. Graphically, the horizontal line $y = 2$ (the horizontal asymptote of the graph) is the only horizontal line that does not meet the graph; every other horizontal line meets it exactly once.

(d) $f: \mathbb{R} \setminus \{3\} \rightarrow \mathbb{R} \setminus \{2\}$ is bijective, and from (c):

$$f^{-1}: \mathbb{R} \setminus \{2\} \rightarrow \mathbb{R} \setminus \{3\}, \quad f^{-1}(y) = \frac{3y+1}{y-2}.$$

Check. $f(f^{-1}(y)) = \frac{2 \cdot \frac{3y+1}{y-2} + 1}{\frac{3y+1}{y-2} - 3} = \frac{\frac{6y+2+y-2}{y-2}}{\frac{3y+1-3y+6}{y-2}} = \frac{7y}{7} = y. \checkmark$

$$f^{-1}(y) = \frac{3y+1}{y-2}, \quad y \neq 2$$

B6. Let $f(x) = \log_3(x^2 - 1)$.

- (a) Find the natural domain D of f .
- (b) Prove that f is not injective on D .
- (c) Prove that $f: D \rightarrow \mathbb{R}$ is surjective.
- (d) Let g be the restriction of f to $(1, +\infty)$. Prove that $g: (1, +\infty) \rightarrow \mathbb{R}$ is bijective and compute g^{-1} .

Solution.

(a) The argument must be positive: $x^2 - 1 > 0 \iff x^2 > 1 \iff x < -1$ or $x > 1$. So

$$D = (-\infty, -1) \cup (1, +\infty).$$

(b) $f(-2) = \log_3 3 = 1$ and $f(2) = \log_3 3 = 1$, with $-2 \neq 2$. So f is not injective. More generally $f(-x) = f(x)$ for every $x \in D$ (f is even): the graph is symmetric with respect to the y -axis, and every horizontal line meets it twice.

(c) Let $y \in \mathbb{R}$ and solve $f(x) = y$. By definition of logarithm,

$$\log_3(x^2 - 1) = y \implies x^2 - 1 = 3^y \implies x^2 = 1 + 3^y.$$

Since $3^y > 0$, we have $1 + 3^y > 1$, so the equation has the two solutions $x = \pm\sqrt{1 + 3^y}$, and both satisfy $|x| > 1$, i.e. belong to D . Hence every $y \in \mathbb{R}$ is taken: $f: D \rightarrow \mathbb{R}$ is surjective.

(d) g is injective. Let $x_1, x_2 > 1$ with $g(x_1) = g(x_2)$. By the injectivity of $\log_3$,

$$x_1^2 - 1 = x_2^2 - 1 \implies x_1^2 = x_2^2 \implies x_1 = x_2,$$

where the last step uses $x_1, x_2 > 0$.

g is surjective onto $\mathbb{R}$. By (c), for every $y \in \mathbb{R}$ the value $x = \sqrt{1 + 3^y}$ satisfies $x > 1$ and $g(x) = y$.

Hence $g: (1, +\infty) \rightarrow \mathbb{R}$ is bijective, and its inverse is obtained by choosing the positive solution in (c):

$$g^{-1}: \mathbb{R} \rightarrow (1, +\infty), \quad g^{-1}(y) = \sqrt{1 + 3^y}.$$

Check. $g(g^{-1}(y)) = \log_3((1 + 3^y) - 1) = \log_3 3^y = y$. ✓

$g^{-1}(y) = \sqrt{1 + 3^y}, \quad y \in \mathbb{R}$
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