

Injective, Surjective and Bijective Functions

Natural domain, inverse function and fundamental examples

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1 Real functions and natural domain

Definition 1 (Function). Let $A, B \subseteq \mathbb{R}$. A *function* $f: A \rightarrow B$ is a rule that assigns to *every* $x \in A$ *exactly one* element $f(x) \in B$.

- A is the *domain* of f ;
- B is the *codomain* of f ;
- the *image* (or *range*) of f is the set of values actually taken:

$$\text{Im } f = f(A) = \{ f(x) : x \in A \} \subseteq B.$$

The *graph* of f is the set of points $\{(x, f(x)) : x \in A\}$ of the Cartesian plane.

Remark. A subset of the plane is the graph of a function of x if and only if every *vertical* line $x = c$ meets it *at most once*. The *horizontal* lines, as we will see, describe injectivity and surjectivity.

Definition 2 (Natural domain, or field of existence). When a function is given only by a formula $y = f(x)$, its *natural domain* (also called *field of existence*) is the largest subset of $\mathbb{R}$ on which the formula makes sense, i.e. on which all the operations it contains can be performed.

Rules for the natural domain

Expression	Condition
$\frac{N(x)}{D(x)}$	$D(x) \neq 0$
$\sqrt[n]{g(x)}$, n even	$g(x) \geq 0$
$\sqrt[n]{g(x)}$, n odd	no condition (beyond g being defined)
$\log_a g(x)$ ($a > 0$, $a \neq 1$)	$g(x) > 0$
$a^{g(x)}$ ($a > 0$)	no condition (beyond g being defined)
polynomials	no condition: domain $\mathbb{R}$

When several operations appear, all the conditions must hold at the same time: the natural domain is the solution set of the *system* of conditions.

Example 1. $f(x) = \frac{\sqrt{x-1}}{x-3}$. Conditions:

$$\begin{cases} x-1 \geq 0 \\ x-3 \neq 0 \end{cases} \implies \begin{cases} x \geq 1 \\ x \neq 3 \end{cases} \quad \text{so} \quad D = [1, 3) \cup (3, +\infty).$$

Example 2. $f(x) = \log_2(4 - x^2)$. Condition: $4 - x^2 > 0 \iff x^2 < 4 \iff -2 < x < 2$, so $D = (-2, 2)$.

Example 3. $f(x) = \sqrt{\log_3 x}$. Conditions:

$$\begin{cases} x > 0 & (\text{argument of the logarithm}) \\ \log_3 x \geq 0 & (\text{radicand}) \end{cases} \implies \begin{cases} x > 0 \\ x \geq 3^0 = 1 \end{cases} \quad \text{so} \quad D = [1, +\infty).$$

2 Injective, surjective and bijective functions

Throughout this section $f: A \rightarrow B$ is a function, with $A, B \subseteq \mathbb{R}$.

2.1 Injective functions

Definition 3 (Injective function). f is *injective* if distinct inputs always give distinct outputs:

$$\forall x_1, x_2 \in A: \quad x_1 \neq x_2 \implies f(x_1) \neq f(x_2).$$

Equivalently (contrapositive), and this is the form used in computations:

$$\forall x_1, x_2 \in A: \quad f(x_1) = f(x_2) \implies x_1 = x_2.$$

How to use it.

- To *prove* that f is injective: assume $f(x_1) = f(x_2)$ and deduce $x_1 = x_2$.
- To *prove* that f is *not* injective: exhibit two *specific* points $x_1 \neq x_2$ with $f(x_1) = f(x_2)$ (a counterexample).

Proposition 1 (Graphical criterion). f is *injective if and only if every horizontal line $y = k$ meets the graph of f at most once*.

Indeed, the intersections of the line $y = k$ with the graph are the points (x, k) with $f(x) = k$; two intersections would give $x_1 \neq x_2$ with $f(x_1) = f(x_2) = k$.

Proposition 2 (Monotonicity). *If f is strictly increasing or strictly decreasing on A , then f is injective.*

Proof. Let f be strictly increasing and $x_1 \neq x_2$, say $x_1 < x_2$. Then $f(x_1) < f(x_2)$, so $f(x_1) \neq f(x_2)$. The decreasing case is identical. $\square$

Remark. The converse is false: $f(x) = \frac{1}{x}$ on $\mathbb{R} \setminus \{0\}$ is injective (if $\frac{1}{x_1} = \frac{1}{x_2}$ then $x_1 = x_2$), but it is not monotone on its whole domain: $-1 < 1$ and $f(-1) = -1 < 1 = f(1)$, while $1 < 2$ and $f(1) = 1 > \frac{1}{2} = f(2)$.

2.2 Surjective functions

Definition 4 (Surjective function). $f: A \rightarrow B$ is *surjective* (onto B) if every element of the codomain is taken at least once:

$$\forall y \in B \quad \exists x \in A: \quad f(x) = y.$$

Equivalently, $\text{Im } f = B$.

How to use it.

- To *prove* that f is surjective: take an arbitrary $y \in B$ and solve the equation $f(x) = y$ for x , checking that the solution belongs to A .
- To *prove* that f is *not* surjective: exhibit a *specific* $y \in B$ for which $f(x) = y$ has no solution in A .

Remark. Surjectivity depends on the codomain. The formula $f(x) = x^2$ gives a function $\mathbb{R} \rightarrow \mathbb{R}$ that is not surjective, and a function $\mathbb{R} \rightarrow [0, +\infty)$ that is surjective. Every function becomes surjective if we replace its codomain B with its image $\text{Im } f$.

Proposition 3 (Graphical criterion). $f: A \rightarrow B$ is surjective if and only if every horizontal line $y = k$ with $k \in B$ meets the graph of f **at least once**.

2.3 Bijective functions

Definition 5 (Bijective function). $f: A \rightarrow B$ is *bijective* if it is both injective and surjective, that is

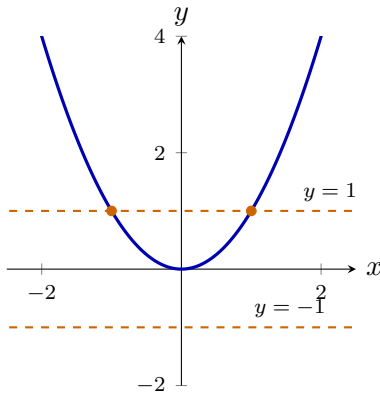
$$\forall y \in B \quad \exists! x \in A : f(x) = y$$

(for every y in the codomain the equation $f(x) = y$ has *exactly one* solution in A).

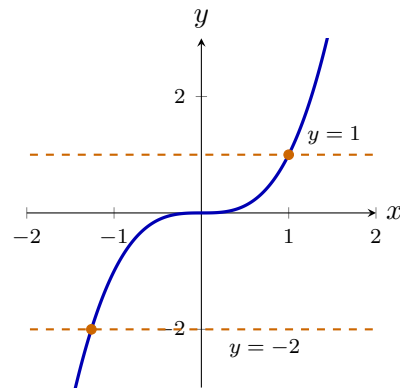
Proposition 4 (Graphical criterion). $f: A \rightarrow B$ is bijective if and only if every horizontal line $y = k$ with $k \in B$ meets the graph of f **exactly once**, and the lines $y = k$ with $k \notin B$ do not meet it.

Summary of the horizontal line test

Property	Algebraic condition	Line $y = k$ meets the graph
injective	$f(x) = k$ has <i>at most one</i> solution	at most once, for every k
surjective onto B	$f(x) = k$ has <i>at least one</i> solution	at least once, for every $k \in B$
bijective onto B	$f(x) = k$ has <i>exactly one</i> solution	exactly once, for every $k \in B$



(a) $y = x^2$ on $\mathbb{R}$: the line $y = 1$ meets the graph twice (not injective), the line $y = -1$ never (not surjective onto $\mathbb{R}$).



(b) $y = x^3$ on $\mathbb{R}$: every horizontal line meets the graph exactly once (bijective $\mathbb{R} \rightarrow \mathbb{R}$).

Figure 1: The horizontal line test.

3 The inverse function

Definition 6 (Inverse function). Let $f: A \rightarrow B$ be bijective. The *inverse function* of f is the function

$$f^{-1}: B \rightarrow A, \quad f^{-1}(y) = \text{the unique } x \in A \text{ such that } f(x) = y.$$

In other words,

$$f^{-1}(y) = x \iff f(x) = y.$$

Only bijective functions have an inverse: injectivity guarantees that x is *unique*, surjectivity that it *exists* for every $y \in B$. The domain of f^{-1} is the codomain (= image) of f , and the image of f^{-1} is the domain of f .

Proposition 5 (Properties of the inverse). *Let $f: A \rightarrow B$ be bijective. Then*

- (a) $f^{-1}(f(x)) = x$ for every $x \in A$, and $f(f^{-1}(y)) = y$ for every $y \in B$;
- (b) f^{-1} is bijective and $(f^{-1})^{-1} = f$;
- (c) if f is strictly increasing (decreasing), so is f^{-1} ;
- (d) the graph of f^{-1} is the reflection of the graph of f in the line $y = x$ (the bisector of the first and third quadrants): the point (a, b) lies on the graph of f if and only if the point (b, a) lies on the graph of f^{-1} .

How to compute the inverse

1. Check that $f: A \rightarrow B$ is bijective (if it is not, restrict the domain and/or the codomain, see below).
2. Write $y = f(x)$ and solve for x in terms of y : $x = g(y)$. The solution must be unique and lie in A .
3. Then $f^{-1}(y) = g(y)$. If desired, rename the variable and write $f^{-1}(x) = g(x)$, so that f and f^{-1} can be drawn in the same plane.

Example 4. $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = 2x + 3$. It is strictly increasing, hence injective; for every $y \in \mathbb{R}$

$$y = 2x + 3 \iff x = \frac{y - 3}{2},$$

so it is surjective and $f^{-1}(y) = \frac{y - 3}{2}$. Check: $f(f^{-1}(y)) = 2 \cdot \frac{y - 3}{2} + 3 = y$.

Restrictions

A function that is not bijective can be made bijective:

- replacing the codomain with the image makes it *surjective*;
- restricting the domain to a set on which it is injective (for instance an interval where it is strictly monotone) makes it *injective*.

The inverse is then the inverse of the *restricted* function. This is exactly what happens for $y = x^2$, see Section 4.2.

4 Fundamental examples

4.1 The cube: $f(x) = x^3$

- **Natural domain:** $\mathbb{R}$ (polynomial).

- **Injective.** Assume $x_1^3 = x_2^3$. Then

$$0 = x_1^3 - x_2^3 = (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2).$$

The second factor can be written as $(x_1 + \frac{x_2}{2})^2 + \frac{3}{4}x_2^2$, a sum of squares, which is 0 only when $x_1 = x_2 = 0$. In every case $x_1 = x_2$. (Alternatively: x^3 is strictly increasing.)

- **Surjective onto $\mathbb{R}$.** For every $y \in \mathbb{R}$, the equation $x^3 = y$ has the solution $x = \sqrt[3]{y}$ (the cube root of a real number always exists).
- **Bijective $\mathbb{R} \rightarrow \mathbb{R}$,** with inverse $f^{-1}(y) = \sqrt[3]{y}$, also bijective $\mathbb{R} \rightarrow \mathbb{R}$.
- **Graph:** every horizontal line meets it exactly once.

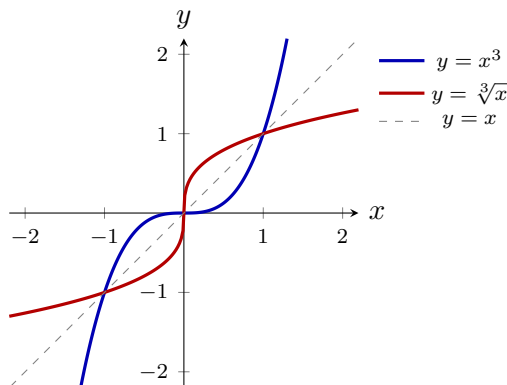


Figure 2: $y = x^3$ and its inverse $y = \sqrt[3]{x}$, symmetric with respect to $y = x$.

4.2 The parabola: $f(x) = x^2$

- **Natural domain:** $\mathbb{R}$. **Image:** $[0, +\infty)$, since $x^2 \geq 0$ and every $y \geq 0$ is $(\sqrt{y})^2$.
- **Not injective on $\mathbb{R}$.** Counterexample: $f(-1) = f(1) = 1$, with $-1 \neq 1$. Graphically, every line $y = k$ with $k > 0$ meets the parabola twice, at $x = \pm\sqrt{k}$.
- **Not surjective onto $\mathbb{R}$.** Counterexample: $x^2 = -1$ has no real solution. Graphically, the lines $y = k$ with $k < 0$ do not meet the parabola.
- **Restriction.** On $[0, +\infty)$ the function is strictly increasing: if $0 \leq x_1 < x_2$ then $x_1^2 < x_2^2$. Hence

$$f: [0, +\infty) \rightarrow [0, +\infty), \quad f(x) = x^2$$

is bijective, and solving $y = x^2$ with $x \geq 0$ gives $x = \sqrt{y}$:

$$f^{-1}(y) = \sqrt{y}.$$

Restricting instead to $(-\infty, 0]$ also gives a bijection onto $[0, +\infty)$, whose inverse is $y \mapsto -\sqrt{y}$.

Remark. The same procedure works for any parabola $y = ax^2 + bx + c$: it is injective on each of the two half-lines determined by the vertex abscissa $x_V = -\frac{b}{2a}$, i.e. on $(-\infty, x_V]$ and on $[x_V, +\infty)$, and never on a set containing points on both sides of x_V . For example $y = x^2 - 2x = (x-1)^2 - 1$ is bijective from $[1, +\infty)$ to $[-1, +\infty)$, with inverse $y \mapsto 1 + \sqrt{y+1}$.

4.3 The square root: $f(x) = \sqrt{x}$

- **Natural domain:** $x \geq 0$, i.e. $[0, +\infty)$ (radicand of an even root). **Image:** $[0, +\infty)$.
- **Injective.** If $\sqrt{x_1} = \sqrt{x_2}$, squaring both sides gives $x_1 = x_2$.
- **Surjective onto $[0, +\infty)$.** For every $y \geq 0$, the equation $\sqrt{x} = y$ has the solution $x = y^2 \geq 0$. It is *not* surjective onto $\mathbb{R}$: $\sqrt{x} = -1$ has no solution.
- **Bijective $[0, +\infty) \rightarrow [0, +\infty)$,** with inverse $f^{-1}(y) = y^2, y \geq 0$: the inverse is the *restricted* parabola, not the whole parabola.

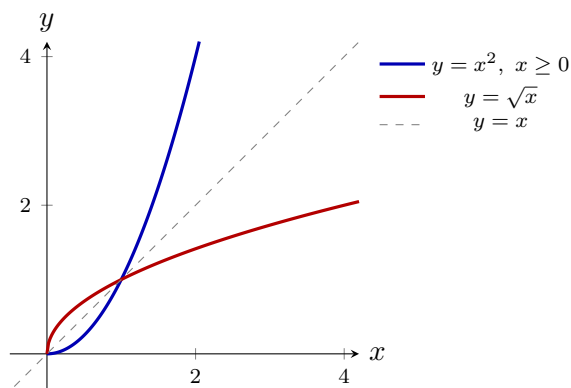


Figure 3: The restricted parabola $y = x^2, x \geq 0$, and its inverse $y = \sqrt{x}$.

4.4 The exponential: $f(x) = a^x, a > 1$

- **Natural domain:** $\mathbb{R}$. **Image:** $(0, +\infty)$, since $a^x > 0$ for every x .
- **Injective.** For $a > 1$ the function a^x is strictly increasing, hence injective. (Equivalently: $a^{x_1} = a^{x_2} \Rightarrow x_1 = x_2$, the property used to solve exponential equations.)
- **Surjective onto $(0, +\infty)$.** For every $y > 0$, the equation $a^x = y$ has the solution $x = \log_a y$. It is *not* surjective onto $\mathbb{R}$: $a^x = 0$ and $a^x = -1$ have no solution, and the horizontal lines $y = k$ with $k \leq 0$ do not meet the graph.
- **Bijective $\mathbb{R} \rightarrow (0, +\infty)$,** with inverse the logarithm:

$$f^{-1}(y) = \log_a y, \quad y > 0.$$

4.5 The logarithm: $f(x) = \log_a x, a > 1$

- **Natural domain:** $x > 0$, i.e. $(0, +\infty)$. **Image:** $\mathbb{R}$.
- **Injective.** For $a > 1$ it is strictly increasing. Algebraically: if $\log_a x_1 = \log_a x_2$, then $x_1 = a^{\log_a x_1} = a^{\log_a x_2} = x_2$.
- **Surjective onto $\mathbb{R}$.** For every $y \in \mathbb{R}$, the equation $\log_a x = y$ has the solution $x = a^y > 0$.
- **Bijective $(0, +\infty) \rightarrow \mathbb{R}$,** with inverse the exponential, $f^{-1}(y) = a^y$.
- **Inverse identities.** Since $\exp_a$ and $\log_a$ are inverse to each other:

$$\log_a(a^x) = x \quad \forall x \in \mathbb{R}, \quad a^{\log_a x} = x \quad \forall x > 0.$$

Remark. For $0 < a < 1$ the functions a^x and $\log_a x$ are strictly *decreasing*, and everything above still holds: they are bijective between the same sets and inverse to each other.

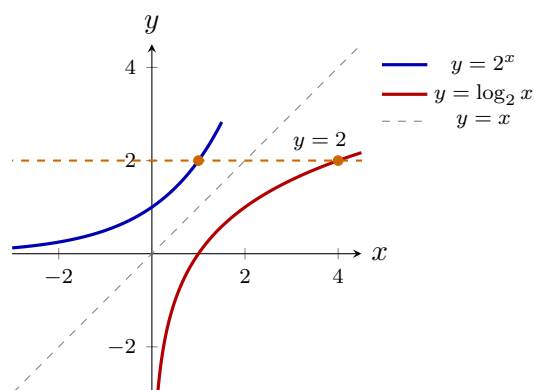


Figure 4: $y = 2^x$ and its inverse $y = \log_2 x$, symmetric with respect to $y = x$. The line $y = 2$ meets each graph exactly once: at $x = 1$ for 2^x and at $x = 4$ for $\log_2 x$. Lines $y = k$ with $k \leq 0$ do not meet the graph of 2^x .

Summary

$f(x)$	Domain	Image	Inj. on domain	Bijjective	Inverse
x^3	$\mathbb{R}$	$\mathbb{R}$	yes	$\mathbb{R} \rightarrow \mathbb{R}$	$\sqrt[3]{y}$
x^2	$\mathbb{R}$	$[0, +\infty)$	no	$[0, +\infty) \rightarrow [0, +\infty)^{(*)}$	$\sqrt{y}$
$\sqrt{x}$	$[0, +\infty)$	$[0, +\infty)$	yes	$[0, +\infty) \rightarrow [0, +\infty)$	$y^2, y \geq 0$
$a^x, a > 1$	$\mathbb{R}$	$(0, +\infty)$	yes	$\mathbb{R} \rightarrow (0, +\infty)$	$\log_a y$
$\log_a x, a > 1$	$(0, +\infty)$	$\mathbb{R}$	yes	$(0, +\infty) \rightarrow \mathbb{R}$	a^y

^(*) after restricting the domain to $[0, +\infty)$.