

Exponential and Logarithmic Equations – Set 2

Exercises with step-by-step solutions

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Solve the following equations (and the system in Exercise 6) in $\mathbb{R}$. The exercises are ordered by increasing difficulty.

1. $9^x - 4 \cdot 3^{x+1} + 27 = 0$

Solution. Write every power in base 3: $9^x = (3^2)^x = (3^x)^2$ and $3^{x+1} = 3 \cdot 3^x$. The equation becomes

$$(3^x)^2 - 12 \cdot 3^x + 27 = 0.$$

Set $t = 3^x$, with $t > 0$:

$$t^2 - 12t + 27 = 0 \implies (t - 3)(t - 9) = 0 \implies t = 3 \text{ or } t = 9.$$

Both values are positive, so both are acceptable. Going back to x :

$$3^x = 3 \implies x = 1, \quad 3^x = 9 = 3^2 \implies x = 2.$$

Check. $x = 1$: $9 - 36 + 27 = 0$. ✓ $x = 2$: $81 - 108 + 27 = 0$. ✓

$x = 1, \quad x = 2$

2. $\log_3(x + 2) + \log_3(x - 4) = 3$

Solution. *Domain.* Both arguments must be positive:

$$x + 2 > 0 \text{ and } x - 4 > 0 \implies x > 4.$$

Solving. For $x > 4$, by the property $\log_a u + \log_a v = \log_a(uv)$,

$$\log_3[(x + 2)(x - 4)] = 3 \implies (x + 2)(x - 4) = 3^3 = 27.$$

Expanding:

$$x^2 - 2x - 8 = 27 \implies x^2 - 2x - 35 = 0 \implies (x - 7)(x + 5) = 0.$$

Hence $x = 7$ or $x = -5$. Only $x = 7$ satisfies $x > 4$; $x = -5$ is rejected.

Check. $\log_3 9 + \log_3 3 = 2 + 1 = 3$. ✓

$x = 7$

3. $2\log_3(x - 1) = \log_3(x + 5)$

Solution. *Domain.* Both arguments must be positive:

$$x - 1 > 0 \text{ and } x + 5 > 0 \implies x > 1.$$

Solving. For $x > 1$ we have $x - 1 > 0$, so the property $n \log_a u = \log_a(u^n)$ can be applied:

$$2 \log_3(x - 1) = \log_3[(x - 1)^2].$$

The equation becomes

$$\log_3[(x - 1)^2] = \log_3(x + 5).$$

The function $\log_3$ is injective, so the arguments must be equal:

$$(x - 1)^2 = x + 5 \implies x^2 - 2x + 1 = x + 5 \implies x^2 - 3x - 4 = 0 \implies (x - 4)(x + 1) = 0.$$

Hence $x = 4$ or $x = -1$. Only $x = 4$ satisfies $x > 1$.

Remark. The value $x = -1$ does satisfy $(x - 1)^2 = x + 5$ (both sides equal 4), but $\log_3(-2)$ is not defined, so $x = -1$ is not a solution of the original equation. This is why the domain must be fixed *before* transforming the equation.

Check. $2 \log_3 3 = 2$ and $\log_3 9 = 2$. ✓

$x = 4$

4. $2^{2x-1} \cdot 3^{x+1} = 216$

Solution. Separate the constant factors from the powers depending on x :

$$2^{2x-1} = \frac{2^{2x}}{2} = \frac{4^x}{2}, \quad 3^{x+1} = 3 \cdot 3^x.$$

The left-hand side becomes

$$\frac{4^x}{2} \cdot 3 \cdot 3^x = \frac{3}{2} (4^x \cdot 3^x) = \frac{3}{2} \cdot 12^x,$$

where we used $a^x \cdot b^x = (ab)^x$. The equation is therefore

$$\frac{3}{2} \cdot 12^x = 216 \implies 12^x = 216 \cdot \frac{2}{3} = 144 = 12^2.$$

The function $x \mapsto 12^x$ is injective, so

$$x = 2.$$

Check. $2^3 \cdot 3^3 = 8 \cdot 27 = 216$. ✓

$x = 2$

5. $\log_3(10 - 3^x) = 2 - x$

Solution. *Domain.* The argument must be positive: $10 - 3^x > 0$, i.e. $3^x < 10$.

Solving. By definition of logarithm, $\log_3 A = B \iff A = 3^B$. Therefore

$$10 - 3^x = 3^{2-x} = \frac{3^2}{3^x} = \frac{9}{3^x}.$$

Set $t = 3^x$, with $t > 0$, and multiply both sides by t :

$$10t - t^2 = 9 \implies t^2 - 10t + 9 = 0 \implies (t - 1)(t - 9) = 0.$$

So $t = 1$ or $t = 9$. Both are positive and both satisfy $t < 10$ (domain condition). Going back to x :

$$3^x = 1 \implies x = 0, \quad 3^x = 9 = 3^2 \implies x = 2.$$

Check. For $x = 0$: $\log_3 9 = 2 = 2 - 0$. ✓ For $x = 2$: $\log_3 1 = 0 = 2 - 2$. ✓

$x = 0, \quad x = 2$

6. Solve the system

$$\begin{cases} 2^{x+y} = 16 \\ \log_3 x + \log_3 y = 1 \end{cases}$$

Solution. *Domain.* The logarithms require $x > 0$ and $y > 0$.

First equation. Since $16 = 2^4$ and $x \mapsto 2^x$ is injective,

$$2^{x+y} = 2^4 \implies x + y = 4.$$

Second equation. For $x > 0, y > 0$, by $\log_a u + \log_a v = \log_a(uv)$,

$$\log_3(xy) = 1 \implies xy = 3^1 = 3.$$

Solving. We need two numbers with sum 4 and product 3. From the first equation $y = 4 - x$; substituting into the second:

$$x(4 - x) = 3 \implies x^2 - 4x + 3 = 0 \implies (x - 1)(x - 3) = 0.$$

- $x = 1 \implies y = 3$;
- $x = 3 \implies y = 1$.

Both pairs have positive components, so both are acceptable.

Check. For $(1, 3)$: $2^4 = 16$ and $\log_3 1 + \log_3 3 = 0 + 1 = 1$. ✓ The pair $(3, 1)$ is checked in the same way, since both equations are symmetric in x and y . ✓

$(x, y) = (1, 3), \quad (x, y) = (3, 1)$
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