

Exponential and Logarithmic Equations

Exercises with step-by-step solutions

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Solve the following equations in $\mathbb{R}$. The exercises are ordered by increasing difficulty. Throughout, $\log x$ denotes the logarithm in base 10.

1. $4^x - 3 \cdot 2^{x+1} + 8 = 0$

Solution. Write every power in base 2: $4^x = (2^2)^x = (2^x)^2$ and $2^{x+1} = 2 \cdot 2^x$. The equation becomes

$$(2^x)^2 - 6 \cdot 2^x + 8 = 0.$$

Set $t = 2^x$, with $t > 0$:

$$t^2 - 6t + 8 = 0 \implies (t - 2)(t - 4) = 0 \implies t = 2 \text{ or } t = 4.$$

Both values are positive, so both are acceptable. Going back to x :

$$2^x = 2 \implies x = 1, \quad 2^x = 4 = 2^2 \implies x = 2.$$

$x = 1, \quad x = 2$

2. $\log_2(x - 1) + \log_2(x + 3) = 5$

Solution. *Domain.* Both arguments must be positive:

$$x - 1 > 0 \text{ and } x + 3 > 0 \implies x > 1.$$

Solving. For $x > 1$, by the property $\log_a u + \log_a v = \log_a(uv)$,

$$\log_2[(x - 1)(x + 3)] = 5 \implies (x - 1)(x + 3) = 2^5 = 32.$$

Expanding:

$$x^2 + 2x - 3 = 32 \implies x^2 + 2x - 35 = 0 \implies (x + 7)(x - 5) = 0.$$

Hence $x = -7$ or $x = 5$. Only $x = 5$ satisfies $x > 1$; $x = -7$ is rejected.

Check. $\log_2 4 + \log_2 8 = 2 + 3 = 5$. ✓

$x = 5$

3. $3^{2x+1} - 10 \cdot 3^x + 3 = 0$

Solution. Since $3^{2x+1} = 3 \cdot 3^{2x} = 3(3^x)^2$, the equation becomes

$$3(3^x)^2 - 10 \cdot 3^x + 3 = 0.$$

Set $t = 3^x$, with $t > 0$:

$$3t^2 - 10t + 3 = 0, \quad t = \frac{10 \pm \sqrt{100 - 36}}{6} = \frac{10 \pm 8}{6},$$

so $t = 3$ or $t = \frac{1}{3}$. Both are positive, hence acceptable:

$$3^x = 3 \implies x = 1, \quad 3^x = \frac{1}{3} = 3^{-1} \implies x = -1.$$

$$\boxed{x = -1, \quad x = 1}$$

4. $2^{x+2} - 2^x = 3^x + 3^{x-1}$

Solution. Collect the common factor on each side:

$$2^{x+2} - 2^x = 2^x(2^2 - 1) = 3 \cdot 2^x, \quad 3^x + 3^{x-1} = 3^x \left(1 + \frac{1}{3}\right) = \frac{4}{3} \cdot 3^x.$$

The equation becomes

$$3 \cdot 2^x = \frac{4}{3} \cdot 3^x.$$

Since $2^x > 0$, we can divide both sides by $\frac{4}{3} \cdot 2^x$:

$$\frac{3^x}{2^x} = \frac{9}{4} \implies \left(\frac{3}{2}\right)^x = \left(\frac{3}{2}\right)^2.$$

The function $x \mapsto \left(\frac{3}{2}\right)^x$ is injective (strictly increasing), so the exponents must be equal:

$$x = 2.$$

Check. $2^4 - 2^2 = 12$ and $3^2 + 3^1 = 12$. ✓

$$\boxed{x = 2}$$

5. $\log_2(9 - 2^x) = 3 - x$

Solution. *Domain.* The argument must be positive: $9 - 2^x > 0$, i.e. $2^x < 9$.

Solving. By definition of logarithm, $\log_2 A = B \iff A = 2^B$. Therefore

$$9 - 2^x = 2^{3-x} = \frac{2^3}{2^x} = \frac{8}{2^x}.$$

Set $t = 2^x$, with $t > 0$, and multiply both sides by t :

$$9t - t^2 = 8 \implies t^2 - 9t + 8 = 0 \implies (t - 1)(t - 8) = 0.$$

So $t = 1$ or $t = 8$. Both are positive and both satisfy $t < 9$ (domain condition). Going back to x :

$$2^x = 1 \implies x = 0, \quad 2^x = 8 = 2^3 \implies x = 3.$$

Check. For $x = 0$: $\log_2 8 = 3 = 3 - 0$. ✓ For $x = 3$: $\log_2 1 = 0 = 3 - 3$. ✓

$$\boxed{x = 0, \quad x = 3}$$

6. $x^{\log x} = 100x$

Solution. *Domain.* $\log x$ is defined only for $x > 0$; for $x > 0$ both sides are positive.

Solving. Since both sides are positive, we may take the base-10 logarithm of both sides (the logarithm is injective, so this gives an equivalent equation):

$$\log(x^{\log x}) = \log(100x).$$

On the left, by $\log(a^b) = b \log a$, we get $\log x \cdot \log x = (\log x)^2$. On the right, by $\log(uv) = \log u + \log v$, we get $\log 100 + \log x = 2 + \log x$. Hence

$$(\log x)^2 = 2 + \log x.$$

Set $y = \log x$ (here y can be any real number):

$$y^2 - y - 2 = 0 \implies (y - 2)(y + 1) = 0 \implies y = 2 \text{ or } y = -1.$$

Going back to x :

$$\log x = 2 \implies x = 10^2 = 100, \quad \log x = -1 \implies x = 10^{-1} = \frac{1}{10}.$$

Both are positive, hence acceptable.

Check. For $x = 100$: $100^2 = 10^4$ and $100 \cdot 100 = 10^4$. ✓ For $x = \frac{1}{10}$: $(10^{-1})^{-1} = 10$ and $100 \cdot \frac{1}{10} = 10$. ✓

$x = \frac{1}{10}, \quad x = 100$
