

Classwork: domains and inverse functions

Solutions

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Exercise I. Find the domain (existence field) of

$$F: \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto \log_3(5^x - 1).$$

Solution. We look for the real numbers x for which the expression $\log_3(5^x - 1)$ makes sense.

Step 1: conditions. The exponential 5^x is defined for every $x \in \mathbb{R}$, so it gives no condition. The logarithm $\log_3(t)$ is defined if and only if its argument is strictly positive, $t > 0$. Hence the only condition is

$$5^x - 1 > 0.$$

Step 2: solve the inequality.

$$5^x - 1 > 0 \iff 5^x > 1 \iff 5^x > 5^0.$$

The base 5 is greater than 1, so the function $x \mapsto 5^x$ is strictly increasing and the inequality between powers is equivalent to the inequality between exponents:

$$5^x > 5^0 \iff x > 0.$$

Step 3: conclusion. The domain of F is

$$\boxed{\{x \in \mathbb{R} \mid x > 0\} = (0, +\infty)}$$

Check. For $x = 1$: $\log_3(5 - 1) = \log_3 4$, which is defined. For $x = 0$: $\log_3(1 - 1) = \log_3 0$, not defined. For $x = -1$: $\log_3(\frac{1}{5} - 1) = \log_3(-\frac{4}{5})$, not defined. $\square$

Exercise II. Find the inverse of

$$F: \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto -2x + 3.$$

Solution. **Step 1: F is bijective.** We fix $y \in \mathbb{R}$ and solve the equation $F(x) = y$ in the unknown x :

$$-2x + 3 = y \iff -2x = y - 3 \iff x = \frac{y - 3}{-2} = \frac{3 - y}{2}.$$

For every $y \in \mathbb{R}$ there is exactly one solution $x = \frac{3 - y}{2} \in \mathbb{R}$. So F is surjective (a solution always exists) and injective (the solution is unique), that is, F is bijective and has an inverse.

Step 2: the inverse. The inverse sends y to the unique x with $F(x) = y$; renaming the variable,

$$\boxed{F^{-1}: \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto \frac{3 - x}{2}}$$

Step 3: check.

$$F(F^{-1}(x)) = -2 \cdot \frac{3 - x}{2} + 3 = -(3 - x) + 3 = x, \quad F^{-1}(F(x)) = \frac{3 - (-2x + 3)}{2} = \frac{2x}{2} = x.$$

Both compositions are the identity, as required. $\square$

Exercise III. Find the inverse, where it exists, of

$$F: \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto 3^{x+1} + 2.$$

Solution. **Step 1: domain.** The exponential 3^{x+1} is defined for every $x \in \mathbb{R}$, so F is defined on all of $\mathbb{R}$.

Step 2: solve $F(x) = y$. We fix $y \in \mathbb{R}$ and look for the $x \in \mathbb{R}$ such that

$$3^{x+1} + 2 = y \iff 3^{x+1} = y - 2.$$

Since $3^{x+1} > 0$ for every x , the equation has no solution if $y - 2 \leq 0$, that is, if $y \leq 2$.

If $y > 2$ we take $\log_3$ of both sides (both are positive):

$$3^{x+1} = y - 2 \iff x + 1 = \log_3(y - 2) \iff x = \log_3(y - 2) - 1.$$

So for every $y > 2$ there is exactly one solution.

Step 3: injectivity and image.

- F is injective: every y has at most one preimage (equivalently, $x \mapsto 3^{x+1}$ is strictly increasing, since $3 > 1$, and adding 2 preserves this).
- The image of F is $\text{Im } F = (2, +\infty)$: the values $y > 2$ are reached, the values $y \leq 2$ are not.
- Hence $F: \mathbb{R} \rightarrow \mathbb{R}$ is *not* surjective, so it is not invertible as a function $\mathbb{R} \rightarrow \mathbb{R}$. It becomes bijective if we restrict the codomain to its image:

$$F: \mathbb{R} \rightarrow (2, +\infty).$$

Step 4: the inverse. From Step 2, renaming the variable,

$$\boxed{F^{-1}: (2, +\infty) \rightarrow \mathbb{R}, \quad x \mapsto \log_3(x - 2) - 1}$$

Note that the domain of $\log_3(x - 2) - 1$ is exactly $x - 2 > 0$, that is, $x > 2$, which matches the image of F .

Step 5: check. For $x \in \mathbb{R}$:

$$F^{-1}(F(x)) = \log_3(3^{x+1} + 2 - 2) - 1 = \log_3(3^{x+1}) - 1 = (x + 1) - 1 = x.$$

For $x > 2$:

$$F(F^{-1}(x)) = 3^{\log_3(x-2)-1+1} + 2 = 3^{\log_3(x-2)} + 2 = (x - 2) + 2 = x.$$

Both compositions are the identity, as required. □