

Logarithms

Rules and solved exercises

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1 Definition

Let $a > 0$, $a \neq 1$ (the *base*) and $x > 0$ (the *argument*). The logarithm of x in base a is the exponent to which a must be raised to give x :

$$\log_a x = y \iff a^y = x.$$

The logarithm of zero or of a negative number is not defined. Notation: $\ln x = \log_e x$ (natural logarithm), $\log_{10} x$ (decimal logarithm).

Immediate consequences. For every admissible a , every $x > 0$ and every real y :

$$(L0) \quad \log_a 1 = 0, \quad \log_a a = 1, \quad (L1) \quad a^{\log_a x} = x, \quad \log_a(a^y) = y.$$

So $x \mapsto a^x$ and $x \mapsto \log_a x$ are inverse functions. The logarithm is increasing if $a > 1$ and decreasing if $0 < a < 1$.

2 Rules

For $x, y > 0$, every real k and admissible bases a, b :

$$(L2) \quad \log_a(xy) = \log_a x + \log_a y, \quad (L3) \quad \log_a \frac{x}{y} = \log_a x - \log_a y, \quad (L4) \quad \log_a(x^k) = k \log_a x,$$

$$(L5) \quad \log_b x = \frac{\log_a x}{\log_a b} \quad (\text{change of base}).$$

They are the exponent rules read backwards: for instance (L2) comes from $a^m a^n = a^{m+n}$. Useful special cases of (L3), (L4) and (L5):

$$\log_a \frac{1}{x} = -\log_a x, \quad \log_a \sqrt[n]{x} = \frac{1}{n} \log_a x, \quad \log_b a = \frac{1}{\log_a b},$$
$$\log_{a^k} x = \frac{1}{k} \log_a x \quad (k \neq 0), \quad \log_{1/a} x = -\log_a x.$$

Warning. *There are no rules for sums or for products of logarithms: $\log_a(x+y) \neq \log_a x + \log_a y$, $\log_a x \cdot \log_a y \neq \log_a(xy)$, $\frac{\log_a x}{\log_a y} \neq \log_a \frac{x}{y}$. Moreover (L4) requires $x > 0$: for $x \neq 0$ one has $\log_a(x^2) = 2 \log_a |x|$, not $2 \log_a x$. When solving an equation, always check that the solutions make every argument positive.*

3 Exercises

Compute or simplify. Letters denote positive real numbers unless stated otherwise.

Exercise 1. $\log_2 32$.

Solution. $32 = 2^5$, so by (L1) $\log_2 32 = \log_2(2^5) = 5$.

Exercise 2. $\log_3 \frac{1}{81}$.

Solution. $\frac{1}{81} = \frac{1}{3^4} = 3^{-4}$, so $\log_3 \frac{1}{81} = -4$.

Exercise 3. $\log_{10} 0.001$.

Solution. $0.001 = 10^{-3}$, so $\log_{10} 0.001 = -3$.

Exercise 4. $\log_5 1 + \log_7 7$.

Solution. By (L0), $\log_5 1 = 0$ and $\log_7 7 = 1$, so the sum is 1.

Exercise 5. $2^{\log_2 7} + \ln(e^3)$.

Solution. By (L1), $2^{\log_2 7} = 7$ and $\ln(e^3) = 3$, so the result is 10.

Exercise 6. $\log_4 8$.

Solution. Write both numbers as powers of 2: we look for y with $4^y = 8$, that is $2^{2y} = 2^3$, so $2y = 3$ and $\log_4 8 = \frac{3}{2}$.

Exercise 7. $\log_9 \sqrt{3}$.

Solution. $9^y = \sqrt{3}$ means $3^{2y} = 3^{1/2}$, so $2y = \frac{1}{2}$ and $\log_9 \sqrt{3} = \frac{1}{4}$.

Exercise 8. $\log_6 4 + \log_6 9$.

Solution. By (L2), $\log_6 4 + \log_6 9 = \log_6 36 = \log_6(6^2) = 2$.

Exercise 9. $\log_2 40 - \log_2 5$.

Solution. By (L3), $\log_2 40 - \log_2 5 = \log_2 \frac{40}{5} = \log_2 8 = 3$.

Exercise 10. $\log_3 \sqrt[5]{27}$.

Solution. By (L4), $\log_3 \sqrt[5]{27} = \log_3(27^{1/5}) = \frac{1}{5} \log_3 27 = \frac{1}{5} \cdot 3 = \frac{3}{5}$.

Exercise 11. $2 \log_5 10 - \log_5 4$.

Solution. By (L4), $2 \log_5 10 = \log_5 100$; then by (L3), $\log_5 100 - \log_5 4 = \log_5 25 = 2$.

Exercise 12. Expand $\log_a \frac{x^3 \sqrt{y}}{z^2}$.

Solution. By (L3), (L2) and (L4): $\log_a \frac{x^3 \sqrt{y}}{z^2} = \log_a x^3 + \log_a y^{1/2} - \log_a z^2 = 3 \log_a x + \frac{1}{2} \log_a y - 2 \log_a z$.

Exercise 13. Write $\ln x + 2 \ln y - \frac{1}{3} \ln z$ as a single logarithm.

Solution. By (L4), $2 \ln y = \ln y^2$ and $\frac{1}{3} \ln z = \ln \sqrt[3]{z}$. By (L2) and (L3), $\ln x + \ln y^2 - \ln \sqrt[3]{z} = \ln \frac{xy^2}{\sqrt[3]{z}}$.

Exercise 14. $\log_8 32$.

Solution. By (L5) with base 2: $\log_8 32 = \frac{\log_2 32}{\log_2 8} = \frac{5}{3}$. Check: $8^{5/3} = (\sqrt[3]{8})^5 = 2^5 = 32$.

Exercise 15. $\log_2 3 \cdot \log_3 16$.

Solution. By (L5), $\log_3 16 = \frac{\log_2 16}{\log_2 3}$, so $\log_2 3 \cdot \log_3 16 = \log_2 3 \cdot \frac{\log_2 16}{\log_2 3} = \log_2 16 = 4$.

Exercise 16. $\log_{1/2} 8$.

Solution. $\log_{1/2} 8 = -\log_2 8 = -3$. Check: $\left(\frac{1}{2}\right)^{-3} = 2^3 = 8$.

Exercise 17. Knowing that $\log_a b = 3$ (with $a, b \neq 1$), compute $\log_b a$ and $\log_{a^2}(b^3)$.

Solution. $\log_b a = \frac{1}{\log_a b} = \frac{1}{3}$. Moreover $\log_{a^2}(b^3) = \frac{1}{2} \log_a(b^3) = \frac{1}{2} \cdot 3 \log_a b = \frac{1}{2} \cdot 9 = \frac{9}{2}$.

Exercise 18. Compute $\log_3((-9)^2)$. Is it equal to $2 \log_3(-9)$?

Solution. $(-9)^2 = 81 = 3^4$, so $\log_3((-9)^2) = 4$. The expression $2 \log_3(-9)$ is not defined, because the argument is negative: (L4) cannot be applied. The correct formula is $\log_3((-9)^2) = 2 \log_3 |-9| = 2 \log_3 9 = 2 \cdot 2 = 4$.

Exercise 19. Solve $\log_3(x-1) = 2$.

Solution. The equation requires $x-1 > 0$, that is $x > 1$. By definition, $\log_3(x-1) = 2$ means $x-1 = 3^2 = 9$, so $x = 10$, which satisfies $x > 1$.

Exercise 20. Solve $\log_2 x + \log_2(x-2) = 3$.

Solution. The arguments must be positive: $x > 0$ and $x-2 > 0$, so $x > 2$. For such x , by (L2) the equation becomes $\log_2(x(x-2)) = 3$, that is $x(x-2) = 2^3 = 8$, or $x^2 - 2x - 8 = 0$. Its roots are $x = 4$ and $x = -2$. Only $x = 4$ satisfies $x > 2$, so the solution is $x = 4$. Check: $\log_2 4 + \log_2 2 = 2 + 1 = 3$.