

Integer and rational exponents

Rules and solved exercises

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1 Integer exponents

For a real number a and a positive integer n ,

$$a^n = \underbrace{a \cdot a \cdots a}_{n \text{ times}}, \quad a^0 = 1 \ (a \neq 0), \quad a^{-n} = \frac{1}{a^n} \ (a \neq 0).$$

For all integers m, n and all real a, b (nonzero whenever a zero or negative exponent appears):

$$\begin{aligned} \text{(P1)} \quad a^m a^n &= a^{m+n}, & \text{(P2)} \quad \frac{a^m}{a^n} &= a^{m-n}, & \text{(P3)} \quad (a^m)^n &= a^{mn}, \\ \text{(P4)} \quad (ab)^n &= a^n b^n, & \text{(P5)} \quad \left(\frac{a}{b}\right)^n &= \frac{a^n}{b^n}, & \text{(P6)} \quad \left(\frac{a}{b}\right)^{-n} &= \left(\frac{b}{a}\right)^n. \end{aligned}$$

Sign of a power. If $a < 0$, then $a^n > 0$ for n even and $a^n < 0$ for n odd. Note that $-2^4 = -(2^4) = -16$, while $(-2)^4 = 16$.

Warning. There is no rule for sums: in general $(a+b)^n \neq a^n + b^n$ and $a^m + a^n \neq a^{m+n}$.

2 Roots and rational exponents

Roots. Let $n \geq 2$ be an integer.

- If n is even and $a \geq 0$, $\sqrt[n]{a}$ is the unique $b \geq 0$ with $b^n = a$. For $a < 0$ it is not defined (in $\mathbb{R}$).
 - If n is odd, $\sqrt[n]{a}$ is the unique real b with $b^n = a$, for every real a ; for example $\sqrt[3]{-8} = -2$.
- In particular $\sqrt[n]{a^n} = |a|$ for n even and $\sqrt[n]{a^n} = a$ for n odd: for instance $\sqrt{x^2} = |x|$.

Rational exponents. For $a > 0$, $m \in \mathbb{Z}$ and $n \geq 1$ an integer,

$$a^{1/n} = \sqrt[n]{a}, \quad a^{m/n} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}.$$

The value does not depend on how the fraction is written ($a^{2/4} = a^{1/2}$), and the rules (P1)–(P6) hold unchanged for rational exponents, as long as the bases are positive.

Rules for roots ($a, b \geq 0$; they are (P1)–(P5) rewritten):

$$\sqrt[n]{a} \sqrt[n]{b} = \sqrt[n]{ab}, \quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}} \ (b > 0), \quad \sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}, \quad \sqrt[n]{a^n b} = a \sqrt[n]{b}.$$

Warning. Rational exponents are defined here only for positive bases. With a negative base the rules break down: if we set $(-8)^{1/3} = -2$, then, since $\frac{1}{3} = \frac{2}{6}$, we should also have $(-8)^{2/6} = -2$; but $\sqrt[6]{(-8)^2} = \sqrt[6]{64} = 2$. For negative numbers use the odd root notation $\sqrt[3]{-8}$ instead. Moreover $\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}$ in general.

3 Exercises

Compute or simplify. Letters denote positive real numbers unless stated otherwise.

Exercise 1. $2^3 \cdot 2^5$.

Solution. By (P1), $2^3 \cdot 2^5 = 2^{3+5} = 2^8 = 256$.

Exercise 2. $\frac{3^7}{3^4}$.

Solution. By (P2), $\frac{3^7}{3^4} = 3^{7-4} = 3^3 = 27$.

Exercise 3. $(5^2)^3$.

Solution. By (P3), $(5^2)^3 = 5^{2 \cdot 3} = 5^6 = 15625$.

Exercise 4. $7^0 + 2^{-3}$.

Solution. $7^0 = 1$ and $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$, so $7^0 + 2^{-3} = 1 + \frac{1}{8} = \frac{9}{8}$.

Exercise 5. $\left(\frac{2}{3}\right)^{-2}$.

Solution. By (P6) and (P5), $\left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^2 = \frac{3^2}{2^2} = \frac{9}{4}$.

Exercise 6. $(-2)^3 \cdot (-2)^{-1}$ and $(-2)^4 - 2^4$.

Solution. By (P1), $(-2)^3 \cdot (-2)^{-1} = (-2)^{3-1} = (-2)^2 = 4$ (even exponent, positive result). For the second, $(-2)^4 = 16$ and $2^4 = 16$, so $(-2)^4 - 2^4 = 0$.

Exercise 7. $(x^3y^{-2})^2 \cdot x^{-4}y^5$, with $x, y \neq 0$.

Solution. By (P4) and (P3), $(x^3y^{-2})^2 = x^6y^{-4}$. Then, by (P1), $x^6y^{-4} \cdot x^{-4}y^5 = x^{6-4}y^{-4+5} = x^2y$.

Exercise 8. $\frac{a^{-2}b^3}{a^3b^{-1}}$, with $a, b \neq 0$.

Solution. By (P2), separately on each base: $\frac{a^{-2}b^3}{a^3b^{-1}} = a^{-2-3}b^{3-(-1)} = a^{-5}b^4 = \frac{b^4}{a^5}$.

Exercise 9. $\frac{6^4}{2^4 \cdot 3^2}$.

Solution. By (P4), $6^4 = (2 \cdot 3)^4 = 2^4 \cdot 3^4$, so $\frac{6^4}{2^4 \cdot 3^2} = \frac{2^4 \cdot 3^4}{2^4 \cdot 3^2} = 3^{4-2} = 9$.

Exercise 10. $16^{1/2} + 27^{1/3}$.

Solution. $16^{1/2} = \sqrt{16} = 4$ and $27^{1/3} = \sqrt[3]{27} = 3$, so the sum is 7.

Exercise 11. $8^{2/3}$.

Solution. $8^{2/3} = (\sqrt[3]{8})^2 = 2^2 = 4$. Equivalently, $\sqrt[3]{8^2} = \sqrt[3]{64} = 4$.

Exercise 12. $32^{-3/5}$.

Solution. $32^{-3/5} = \frac{1}{32^{3/5}} = \frac{1}{(\sqrt[5]{32})^3} = \frac{1}{2^3} = \frac{1}{8}$.

Exercise 13. $\left(\frac{9}{4}\right)^{-3/2}$.

Solution. By (P6), $\left(\frac{9}{4}\right)^{-3/2} = \left(\frac{4}{9}\right)^{3/2} = \left(\sqrt{\frac{4}{9}}\right)^3 = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$.

Exercise 14. Write $\sqrt[3]{x^2} \cdot \sqrt{x}$ as a single power of x .

Solution. $\sqrt[3]{x^2} = x^{2/3}$ and $\sqrt{x} = x^{1/2}$, so by (P1) $x^{2/3} \cdot x^{1/2} = x^{2/3+1/2} = x^{7/6} = \sqrt[6]{x^7} = x\sqrt[6]{x}$.

Exercise 15. $\sqrt{(-5)^2}$ and $\sqrt[3]{(-5)^3}$.

Solution. $\sqrt{(-5)^2} = \sqrt{25} = 5 = |-5|$: an even root is never negative, so the answer is not -5 . Instead $\sqrt[3]{(-5)^3} = \sqrt[3]{-125} = -5$, since $(-5)^3 = -125$ and odd roots keep the sign.

Exercise 16. $\sqrt[3]{\sqrt{64}}$.

Solution. $\sqrt[3]{\sqrt{64}} = \sqrt[6]{64} = 64^{1/6} = (2^6)^{1/6} = 2$. Check: $\sqrt{64} = 8$ and $\sqrt[3]{8} = 2$.

Exercise 17. $\sqrt{12} \cdot \sqrt{3}$.

Solution. Same index, so $\sqrt{12} \cdot \sqrt{3} = \sqrt{12 \cdot 3} = \sqrt{36} = 6$.

Exercise 18. $\sqrt[3]{54} - \sqrt[3]{16}$.

Solution. $54 = 3^3 \cdot 2$ and $16 = 2^3 \cdot 2$, so $\sqrt[3]{54} = 3\sqrt[3]{2}$ and $\sqrt[3]{16} = 2\sqrt[3]{2}$. Hence $\sqrt[3]{54} - \sqrt[3]{16} = 3\sqrt[3]{2} - 2\sqrt[3]{2} = \sqrt[3]{2}$. (Note that $\sqrt[3]{54 - 16} = \sqrt[3]{38}$ would be wrong.)

Exercise 19. $(a^{1/2}b^{-1/3})^6$.

Solution. By (P4) and (P3), $(a^{1/2}b^{-1/3})^6 = a^{6/2}b^{-6/3} = a^3b^{-2} = \frac{a^3}{b^2}$.

Exercise 20. $2^{1/2} \cdot 4^{1/4} \cdot 8^{1/6}$.

Solution. Write every base as a power of 2: $4^{1/4} = (2^2)^{1/4} = 2^{1/2}$ and $8^{1/6} = (2^3)^{1/6} = 2^{1/2}$. Then $2^{1/2} \cdot 2^{1/2} \cdot 2^{1/2} = 2^{3/2} = \sqrt{2^3} = 2\sqrt{2}$.