

Exponential and logarithmic equations

Methods and solved exercises

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Throughout, $a > 0$ with $a \neq 1$. We use two facts: $a^y > 0$ for every real y , and the functions $y \mapsto a^y$ and $x \mapsto \log_a x$ are injective (inverse of each other).

1 Exponential equations

- (E1) **Same base.** $a^{f(x)} = a^{g(x)} \iff f(x) = g(x)$. Write both sides as powers of the same base when possible: $4^x = 8$ becomes $2^{2x} = 2^3$.
- (E2) **Isolated power.** $a^{f(x)} = b$ has no solutions if $b \leq 0$; if $b > 0$ it is equivalent to $f(x) = \log_a b$.
- (E3) **Different bases.** If $a^{f(x)} = b^{g(x)}$ with no common base, take logarithms (in any base, e.g. $\ln$) of both sides: $f(x) \ln a = g(x) \ln b$.
- (E4) **Common factor.** $a^{x+k} = a^k \cdot a^x$, so sums such as $2^{x+1} + 2^x$ can be written as $(2+1)2^x$.
- (E5) **Substitution.** If the equation involves only a^x and $a^{2x} = (a^x)^2$, set $t = a^x$ and solve in t ; then discard the values $t \leq 0$, since $a^x > 0$.

2 Logarithmic equations

- (D) **Domain first.** Before anything else, impose that every argument is positive. At the end, keep only the solutions in the domain.
- (G1) $\log_a f(x) = c \iff f(x) = a^c$ (and then automatically $f(x) > 0$).
- (G2) $\log_a f(x) = \log_a g(x) \iff f(x) = g(x)$, for x in the domain.
- (G3) **Combine first.** Use $\log_a f + \log_a g = \log_a(fg)$, $\log_a f - \log_a g = \log_a \frac{f}{g}$, $k \log_a f = \log_a f^k$ to reduce to (G1) or (G2).
- (G4) **Different bases.** Reduce to a single base with $\log_b x = \frac{\log_a x}{\log_a b}$.
- (G5) **Substitution.** If the equation involves only $\log_a x$ (e.g. $(\log_a x)^2$), set $t = \log_a x$, solve in t , then $x = a^t$.

Warning. The rules of (G3) may enlarge the domain: $\log_a f + \log_a g$ requires $f > 0$ and $g > 0$, while $\log_a(fg)$ only requires $fg > 0$. This is why the domain must be computed on the original equation. Similarly, $\log_a(x^2)$ is defined for $x \neq 0$, while $2 \log_a x$ only for $x > 0$.

3 Exercises

Solve in $\mathbb{R}$.

Exercise 1. $2^x = 32$.

Solution. $32 = 2^5$, so $2^x = 2^5$ and by (E1) $x = 5$.

Exercise 2. $3^{x+1} = 27$.

Solution. $27 = 3^3$, so by (E1) $x + 1 = 3$, that is $x = 2$.

Exercise 3. $4^x = 8$.

Solution. In base 2: $2^{2x} = 2^3$, so $2x = 3$ and $x = \frac{3}{2}$.

Exercise 4. $5^{x^2-3x} = \frac{1}{25}$.

Solution. $\frac{1}{25} = 5^{-2}$, so by (E1) $x^2 - 3x = -2$, that is $x^2 - 3x + 2 = (x - 1)(x - 2) = 0$. The solutions are $x = 1$ and $x = 2$.

Exercise 5. $\left(\frac{1}{3}\right)^x = 9$.

Solution. $\left(\frac{1}{3}\right)^x = 3^{-x}$ and $9 = 3^2$, so $-x = 2$ and $x = -2$.

Exercise 6. $2^x = 5$.

Solution. 5 is not a power of 2 with integer exponent; by (E2), $x = \log_2 5 = \frac{\ln 5}{\ln 2}$.

Exercise 7. $3^{2x-1} = 7$.

Solution. By (E2), $2x - 1 = \log_3 7$, so $x = \frac{1 + \log_3 7}{2}$.

Exercise 8. $2^x = 3^{x-1}$.

Solution. By (E3), taking $\ln$ of both sides: $x \ln 2 = (x - 1) \ln 3$, that is $x(\ln 3 - \ln 2) = \ln 3$. Since $\ln 3 - \ln 2 = \ln \frac{3}{2} \neq 0$,

$$x = \frac{\ln 3}{\ln 3 - \ln 2} = \frac{\ln 3}{\ln \frac{3}{2}}.$$

Exercise 9. $3^x = -9$.

Solution. $3^x > 0$ for every real x , while $-9 < 0$: by (E2) the equation has no solutions.

Exercise 10. $2^{x+1} + 2^x = 24$.

Solution. By (E4), $2^{x+1} + 2^x = 2 \cdot 2^x + 2^x = 3 \cdot 2^x$. So $3 \cdot 2^x = 24$, $2^x = 8 = 2^3$ and $x = 3$.

Exercise 11. $4^x - 3 \cdot 2^x - 4 = 0$.

Solution. $4^x = (2^x)^2$. By (E5), with $t = 2^x > 0$: $t^2 - 3t - 4 = (t - 4)(t + 1) = 0$, so $t = 4$ or $t = -1$. The value $t = -1$ is discarded because $t > 0$. From $2^x = 4$ we get $x = 2$.

Exercise 12. $9^x - 4 \cdot 3^x + 3 = 0$.

Solution. With $t = 3^x > 0$: $t^2 - 4t + 3 = (t - 1)(t - 3) = 0$, so $t = 1$ or $t = 3$. Both are positive: $3^x = 1$ gives $x = 0$ and $3^x = 3$ gives $x = 1$.

Exercise 13. $\log_2 x = 5$.

Solution. Domain: $x > 0$. By (G1), $x = 2^5 = 32$, which is in the domain.

Exercise 14. $\log_3(2x + 1) = 2$.

Solution. Domain: $2x + 1 > 0$, that is $x > -\frac{1}{2}$. By (G1), $2x + 1 = 3^2 = 9$, so $x = 4$, which is in the domain.

Exercise 15. Solve $\log_3(x^2) = 2$ and $2 \log_3 x = 2$, and compare.

Solution. First equation. Domain: $x^2 > 0$, that is $x \neq 0$. By (G1), $x^2 = 9$, so $x = 3$ or $x = -3$: both are solutions.

Second equation. Domain: $x > 0$. Dividing by 2, $\log_3 x = 1$, so $x = 3$ is the only solution.

The two equations are *not* equivalent: $\log_3(x^2) = 2\log_3|x|$, and it equals $2\log_3 x$ only for $x > 0$.

Exercise 16. $\log_3(x+2) = \log_3(2x-1)$.

Solution. Domain: $x+2 > 0$ and $2x-1 > 0$, that is $x > \frac{1}{2}$. By (G2), $x+2 = 2x-1$, so $x = 3$, which is in the domain.

Exercise 17. $\log_2 x + \log_2(x+2) = 3$.

Solution. Domain: $x > 0$ and $x+2 > 0$, that is $x > 0$. By (G3), $\log_2(x(x+2)) = 3$, so by (G1) $x^2 + 2x = 8$, that is $x^2 + 2x - 8 = (x-2)(x+4) = 0$. Of the two roots, $x = -4$ is outside the domain; the only solution is $x = 2$. Check: $\log_2 2 + \log_2 4 = 1 + 2 = 3$.

Exercise 18. $\log_3(x+1) - \log_3(x-1) = 1$.

Solution. Domain: $x+1 > 0$ and $x-1 > 0$, that is $x > 1$. By (G3), $\log_3 \frac{x+1}{x-1} = 1$, so $\frac{x+1}{x-1} = 3$. Multiplying by $x-1 \neq 0$: $x+1 = 3x-3$, so $x = 2$, which is in the domain.

Exercise 19. $(\log_2 x)^2 - 3\log_2 x + 2 = 0$.

Solution. Domain: $x > 0$. By (G5), with $t = \log_2 x$: $t^2 - 3t + 2 = (t-1)(t-2) = 0$, so $t = 1$ or $t = 2$. Hence $x = 2^1 = 2$ or $x = 2^2 = 4$, both in the domain.

Exercise 20. $\log_4 x + \log_2 x = 6$.

Solution. Domain: $x > 0$. By (G4), $\log_4 x = \frac{\log_2 x}{\log_2 4} = \frac{1}{2} \log_2 x$. The equation becomes $\frac{3}{2} \log_2 x = 6$, so $\log_2 x = 4$ and $x = 2^4 = 16$. Check: $\log_4 16 + \log_2 16 = 2 + 4 = 6$.