Classwork

Exponents, logarithms, exponential and logarithmic equations

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Letters denote positive real numbers. Justify every step.

1 Exponents

Exercise 1. Compute $\dfrac{2^5 \cdot 4^{-1}}{8^{2/3}}$.

Solution. Write everything in base 2: $4^{-1} = (2^2)^{-1} = 2^{-2}$ and $8^{2/3} = (2^3)^{2/3} = 2^2$. Then

$$\frac{2^5 \cdot 2^{-2}}{2^2} = 2^{5-2-2} = 2^1 = 2.$$

Exercise 2. Simplify $\left(\dfrac{27x^6}{y^{-3}}\right)^{2/3}$.

Solution. First $\dfrac{27x^6}{y^{-3}} = 27x^6y^3$. Then, by the rules for the power of a product and the power of a power,

$$\left(27x^6y^3\right)^{2/3} = 27^{2/3}\, x^{6\cdot\frac{2}{3}}\, y^{3\cdot\frac{2}{3}} = \left(\sqrt[3]{27}\right)^2 x^4y^2 = 9x^4y^2.$$

Exercise 3. Compute $\sqrt{50} + \sqrt{18} - \sqrt{8}$.

Solution. Take the square factors out of the roots: $\sqrt{50} = \sqrt{25 \cdot 2} = 5\sqrt{2}$, $\sqrt{18} = \sqrt{9 \cdot 2} = 3\sqrt{2}$, $\sqrt{8} = \sqrt{4 \cdot 2} = 2\sqrt{2}$. Hence

$$\sqrt{50} + \sqrt{18} - \sqrt{8} = (5 + 3 - 2)\sqrt{2} = 6\sqrt{2}.$$

Exercise 4. Simplify $\dfrac{\sqrt[3]{a^2} \cdot \sqrt{a}}{\sqrt[6]{a}}$.

Solution. In exponential form the expression is $\dfrac{a^{2/3}\, a^{1/2}}{a^{1/6}} = a^{\frac{2}{3}+\frac{1}{2}-\frac{1}{6}}$. Since $\dfrac{2}{3} + \dfrac{1}{2} - \dfrac{1}{6} = \dfrac{4+3-1}{6} = 1$, the expression equals a.

2 Logarithms

Exercise 5. Compute $\log_2 \dfrac{1}{\sqrt{8}}$.

Solution. $\dfrac{1}{\sqrt{8}} = 8^{-1/2} = (2^3)^{-1/2} = 2^{-3/2}$, so $\log_2 \dfrac{1}{\sqrt{8}} = -\dfrac{3}{2}$.

Exercise 6. Compute $\log_3 54 - \log_3 2 + \log_3 \dfrac{1}{9}$.

Solution. By the quotient and product rules,

$$\log_3 54 - \log_3 2 + \log_3 \frac{1}{9} = \log_3 \left(\frac{54}{2} \cdot \frac{1}{9}\right) = \log_3 \frac{27}{9} = \log_3 3 = 1.$$

Exercise 7. Compute $9^{\log_3 5}$.
Solution. $9^{\log_3 5} = \left(3^2\right)^{\log_3 5} = 3^{2\log_3 5} = 3^{\log_3 25} = 25$, using $k\log_a x = \log_a x^k$ and $a^{\log_a x} = x$.

Exercise 8. Compute $\log_{25} 125 + \log_{1/3} 27$.
Solution. By change of base to base 5: $\log_{25} 125 = \dfrac{\log_5 125}{\log_5 25} = \dfrac{3}{2}$. Moreover $\log_{1/3} 27 = -\log_3 27 = -3$ (check: $\left(\frac{1}{3}\right)^{-3} = 27$). The sum is $\dfrac{3}{2} - 3 = -\dfrac{3}{2}$.

3 Equations

Solve in $\mathbb{R}$.

Exercise 9. $2^{3x-1} = 4^{x+1}$.
Solution. $4^{x+1} = 2^{2(x+1)} = 2^{2x+2}$. Equating the exponents of the same base, $3x - 1 = 2x + 2$, so $x = 3$.

Exercise 10. $4^x - 2^{x+3} + 16 = 0$.
Solution. $4^x = (2^x)^2$ and $2^{x+3} = 8 \cdot 2^x$. With $t = 2^x > 0$ the equation becomes $t^2 - 8t + 16 = (t-4)^2 = 0$, so $t = 4$. From $2^x = 4$ we get $x = 2$.

Exercise 11. $5^x = 3 \cdot 2^x$.
Solution. Dividing by $2^x > 0$: $\left(\dfrac{5}{2}\right)^x = 3$, so

$$x = \log_{5/2} 3 = \frac{\ln 3}{\ln \frac{5}{2}} = \frac{\ln 3}{\ln 5 - \ln 2}.$$

Exercise 12. $3^{x+2} - 3^x = 72$.
Solution. Collect 3^x: $3^{x+2} - 3^x = 9 \cdot 3^x - 3^x = 8 \cdot 3^x$. So $8 \cdot 3^x = 72$, $3^x = 9$ and $x = 2$.

Exercise 13. $\log_2(x+3) + \log_2(x-3) = 4$.
Solution. Domain: $x+3 > 0$ and $x-3 > 0$, that is $x > 3$. For such x, $\log_2\big((x+3)(x-3)\big) = 4$, so $x^2 - 9 = 2^4 = 16$ and $x^2 = 25$, giving $x = \pm 5$. Only $x = 5$ is in the domain. Check: $\log_2 8 + \log_2 2 = 3 + 1 = 4$.

Exercise 14. $\log_5\big(x^2 - 4x\big) = 1$.
Solution. Domain: $x^2 - 4x = x(x-4) > 0$, that is $x < 0$ or $x > 4$. By definition of logarithm, $x^2 - 4x = 5$, that is $x^2 - 4x - 5 = (x-5)(x+1) = 0$, so $x = 5$ or $x = -1$. Both are in the domain: for $x = 5$ the argument is $25 - 20 = 5$, for $x = -1$ it is $1 + 4 = 5$. The solutions are $x = -1$ and $x = 5$.

Exercise 15. $2\log_3 x - \log_3(x+6) = 1$.
Solution. Domain: $x > 0$ and $x + 6 > 0$, that is $x > 0$. For such x, $2\log_3 x - \log_3(x+6) = \log_3 \dfrac{x^2}{x+6}$, so $\dfrac{x^2}{x+6} = 3$, that is $x^2 - 3x - 18 = (x-6)(x+3) = 0$. The root $x = -3$ is outside the domain; the solution is $x = 6$. Check: $2\log_3 6 - \log_3 12 = \log_3 \dfrac{36}{12} = \log_3 3 = 1$.

Exercise 16. $\log_2 x + 3\log_x 2 = 4$.
Solution. Domain: $x > 0$ (argument of $\log_2 x$) and $x \neq 1$ (base of $\log_x 2$). By change of base, $\log_x 2 = \dfrac{1}{\log_2 x}$, and $\log_2 x \neq 0$ because $x \neq 1$. With $t = \log_2 x$ the equation becomes $t + \dfrac{3}{t} = 4$, that is $t^2 - 4t + 3 = (t-1)(t-3) = 0$, so $t = 1$ or $t = 3$. Hence $x = 2$ or $x = 8$, both in the domain. Check: for $x = 2$, $1 + 3 \cdot 1 = 4$; for $x = 8$, $3 + 3 \cdot \dfrac{1}{3} = 4$.