

Math 0 — Classwork

Exponential equations and analytic geometry in the plane Full solutions

Caboara

Exponential equations

Throughout, we use the following basic facts about b^x with $b > 0$, $b \neq 1$:

- $b^x > 0$ for every $x \in \mathbb{R}$;
- $x \mapsto b^x$ is injective, so $b^u = b^v \iff u = v$;
- for $c > 0$, $b^x = c \iff x = \log_b c$.

Exercise I. Solve $5^{x^2-3x} = \frac{1}{25}$.

Solution. Write both sides as powers of 5: since $25 = 5^2$,

$$\frac{1}{25} = 5^{-2}.$$

The equation becomes

$$5^{x^2-3x} = 5^{-2}.$$

Since $x \mapsto 5^x$ is injective, we can equate the exponents:

$$x^2 - 3x = -2 \iff x^2 - 3x + 2 = 0.$$

The discriminant is $\Delta = 9 - 8 = 1$, hence

$$x = \frac{3 \pm 1}{2}, \quad \text{i.e.} \quad x = 1 \quad \text{or} \quad x = 2.$$

Equivalently, $x^2 - 3x + 2 = (x-1)(x-2)$.

Check. $x = 1$: $5^{1-3} = 5^{-2} = \frac{1}{25}$. $x = 2$: $5^{4-6} = 5^{-2} = \frac{1}{25}$.

Solutions: $x = 1$, $x = 2$. □

Exercise II. Solve $\left(\frac{1}{3}\right)^x = 9$.

Solution. Write both sides as powers of 3:

$$\left(\frac{1}{3}\right)^x = (3^{-1})^x = 3^{-x}, \quad 9 = 3^2.$$

Hence $3^{-x} = 3^2$, and equating the exponents,

$$-x = 2 \iff x = -2.$$

Check. $\left(\frac{1}{3}\right)^{-2} = 3^2 = 9$.

Solution: $x = -2$. □

Exercise III. Solve $3^{2x-1} = 7$.

Solution. Here 7 is not a rational power of 3, so we use logarithms. Since $7 > 0$,

$$3^{2x-1} = 7 \iff 2x - 1 = \log_3 7.$$

Solving the linear equation,

$$2x = 1 + \log_3 7 \iff x = \frac{1 + \log_3 7}{2}.$$

The result can be written more compactly. Since $1 = \log_3 3$,

$$1 + \log_3 7 = \log_3 3 + \log_3 7 = \log_3 21, \quad \text{so} \quad x = \frac{\log_3 21}{2} = \log_3 \sqrt{21}.$$

(Equivalently, $x = \log_9 21$, because $\log_9 21 = \frac{\log_3 21}{\log_3 9} = \frac{\log_3 21}{2}$.)

Alternatively: multiply both sides by 3, getting $3^{2x} = 21$, i.e. $9^x = 21$, so $x = \log_9 21$.

Numerical value. $x = \frac{\ln 21}{2 \ln 3} \approx \frac{3.0445}{2.1972} \approx 1.386$.

Solution: $x = \frac{1 + \log_3 7}{2} = \log_9 21$. □

Exercise IV. Solve $3^x = a$, where $a < 0$.

Solution. For every $x \in \mathbb{R}$ we have $3^x > 0$. Since $a < 0$, the left-hand side is always positive and the right-hand side is negative, so they can never be equal:

$$3^x > 0 > a \quad \forall x \in \mathbb{R}.$$

The equation has no real solutions. □

Remark. More generally, $3^x = a$ has exactly one solution $x = \log_3 a$ if $a > 0$, and no solution if $a \leq 0$ (also $a = 0$ is excluded, since $3^x \neq 0$).

Exercise V. Solve $25^{x+1} - 25^x = 0$.

Solution. Use $25^{x+1} = 25^x \cdot 25$ and factor out 25^x :

$$25^{x+1} - 25^x = 25 \cdot 25^x - 25^x = 25^x (25 - 1) = 24 \cdot 25^x.$$

The equation becomes

$$24 \cdot 25^x = 0 \iff 25^x = 0.$$

But $25^x > 0$ for every real x , so this is impossible.

(Equivalently: $25^{x+1} = 25^x$ would force $x + 1 = x$ by injectivity, i.e. $1 = 0$.)

The equation has no real solutions. □

Exercise VI. Consider the equation

$$7 \cdot 125^x + 25^x + a = 0.$$

Find the values $a \neq 0$ for which there is at least one solution.

Solution. *Step 1: common base.* Since $125 = 5^3$ and $25 = 5^2$,

$$125^x = (5^x)^3, \quad 25^x = (5^x)^2.$$

Step 2: substitution. Put $t = 5^x$. As x ranges over $\mathbb{R}$, t ranges over *all* of $(0, +\infty)$, and each $t > 0$ corresponds to exactly one $x = \log_5 t$. The equation becomes

$$7t^3 + t^2 + a = 0 \iff 7t^3 + t^2 = -a, \quad t > 0.$$

So the original equation has a solution in x if and only if the equation $f(t) = -a$ has a solution with $t > 0$, where

$$f(t) = 7t^3 + t^2.$$

Step 3: values taken by f on $(0, +\infty)$.

- For $t > 0$ both $7t^3$ and t^2 are positive, so $f(t) > 0$. Hence if $-a \leq 0$, i.e. $a \geq 0$, there is no solution.
- f is strictly increasing on $(0, +\infty)$: if $0 < t_1 < t_2$ then $t_1^3 < t_2^3$ and $t_1^2 < t_2^2$, so $f(t_1) < f(t_2)$.
- f is continuous, $f(t) \rightarrow 0$ as $t \rightarrow 0^+$ and $f(t) \rightarrow +\infty$ as $t \rightarrow +\infty$.

Therefore f maps $(0, +\infty)$ bijectively onto $(0, +\infty)$: for every $c > 0$ the equation $f(t) = c$ has exactly one solution $t > 0$.

Step 4: conclusion. The equation $f(t) = -a$ has a solution $t > 0$ if and only if $-a > 0$, that is $a < 0$. In that case there is exactly one $t_0 > 0$ with $f(t_0) = -a$, and correspondingly exactly one $x_0 = \log_5 t_0$.

Example. For $a = -8$: $7t^3 + t^2 = 8$ gives $t = 1$ (indeed $7 + 1 = 8$), so $x = \log_5 1 = 0$. Check: $7 \cdot 125^0 + 25^0 - 8 = 7 + 1 - 8 = 0$.

Answer: there is at least one solution if and only if $a < 0$ (and then the solution is unique).

□