

Math 0 – Analytic geometry

Lines, parabolas and circles: solved exercises

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Reminders. A parabola with vertical axis has equation $y = ax^2 + bx + c$, $a \neq 0$; its vertex has $x_V = -\frac{b}{2a}$. The equation $x^2 + y^2 + Dx + Ey + F = 0$ can be written, completing the squares, as

$$\left(x + \frac{D}{2}\right)^2 + \left(y + \frac{E}{2}\right)^2 = \frac{D^2 + E^2}{4} - F.$$

It is a circle, with center $(-\frac{D}{2}, -\frac{E}{2})$ and radius $r = \sqrt{\frac{D^2 + E^2}{4} - F}$, if and only if $\frac{D^2 + E^2}{4} - F > 0$. Two non-vertical lines are perpendicular if and only if the product of their slopes is -1 . The distance between $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Exercise I. Let $\gamma: y = x^2 - 3x$ and $C: x^2 + y^2 - 2x - 15 = 0$. Find $\gamma \cap C$.

Solution. The curves. Completing the square, $C: (x - 1)^2 + y^2 = 16$: circle with center $(1, 0)$ and radius 4. The parabola γ has vertex at $x_V = \frac{3}{2}$, $y_V = \frac{9}{4} - \frac{9}{2} = -\frac{9}{4}$, and meets the x -axis at $x = 0$ and $x = 3$.

The system. The points of $\gamma \cap C$ are the solutions of

$$\begin{cases} y = x^2 - 3x \\ x^2 + y^2 - 2x - 15 = 0. \end{cases}$$

Substituting the first equation into the second:

$$\begin{aligned} x^2 + (x^2 - 3x)^2 - 2x - 15 &= 0 \iff x^4 - 6x^3 + 9x^2 + x^2 - 2x - 15 = 0 \\ &\iff f(x) := x^4 - 6x^3 + 10x^2 - 2x - 15 = 0. \end{aligned}$$

No rational roots. By the rational root test, a rational root must be one of $\pm 1, \pm 3, \pm 5, \pm 15$. We have $f(1) = -12$, $f(-1) = 4$, $f(3) = -12$, $f(-3) = 324$, $f(5) = 100$, $f(-5) = 1620$, and $f(\pm 15) \neq 0$ ($f(15) = 32580$, $f(-15) = 73140$). So f has no rational roots, and the intersection points cannot be found with the usual factoring methods.

There are exactly two real roots. Set $x = u + \frac{3}{2}$. Then $x^2 - 3x = u^2 - \frac{9}{4}$ and $x - 1 = u + \frac{1}{2}$, and since $f(x) = (x^2 - 3x)^2 + (x - 1)^2 - 16$,

$$f = \left(u^2 - \frac{9}{4}\right)^2 + \left(u + \frac{1}{2}\right)^2 - 16 = u^4 - \frac{7}{2}u^2 + u - \frac{171}{16} =: g(u).$$

- If $u^2 \leq \frac{7}{2}$, then $u^4 - \frac{7}{2}u^2 = u^2(u^2 - \frac{7}{2}) \leq 0$ and $u < 2$, so $g(u) \leq u - \frac{171}{16} < 0$: no roots there.
- If $u > \sqrt{7/2}$, then $g'(u) = 4u^3 - 7u + 1 = u(4u^2 - 7) + 1 > 0$: g is increasing, from a negative value to $+\infty$, so it has exactly one root.
- If $u < -\sqrt{7/2}$, then $u < 0$ and $4u^2 - 7 > 0$, so $g'(u) = u(4u^2 - 7) + 1 < 0$ (because $u(4u^2 - 7) < -\sqrt{7/2} \cdot 7 < -1$): g is decreasing, from $+\infty$ to a negative value, so it has exactly one root.

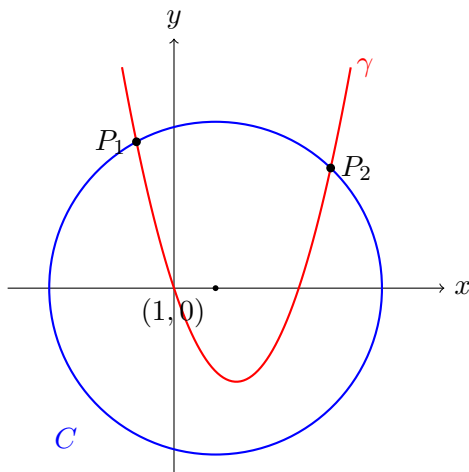
Location and approximate values. From $f(-1) = 4 > 0$, $f(0) = -15 < 0$, $f(3) = -12 < 0$, $f(4) = 9 > 0$, the two roots lie in $(-1, 0)$ and in $(3, 4)$. Bisection (or Newton's method) gives

$$x_1 \approx -0.902, \quad x_2 \approx 3.767,$$

and from $y = x^2 - 3x$,

$$\gamma \cap C = \{P_1, P_2\}, \quad P_1 \approx (-0.902, 3.519), \quad P_2 \approx (3.767, 2.889).$$

Geometrically: the vertex $(\frac{3}{2}, -\frac{9}{4})$ lies inside C , since $(\frac{1}{2})^2 + (\frac{9}{4})^2 = \frac{85}{16} < 16$, so the parabola enters the disc and leaves it once on each side.



Exercise II. Let $\gamma: y = x^2 - 2x - 4$ and $C: x^2 + y^2 - 2x - 24 = 0$. Find $\gamma \cap C$ and draw γ and C .

Solution. The system.

$$\begin{cases} y = x^2 - 2x - 4 \\ x^2 + y^2 - 2x - 24 = 0. \end{cases}$$

Instead of substituting y (which gives a fourth degree equation), note that both equations contain $x^2 - 2x$. From the first, $x^2 - 2x = y + 4$. Substituting into the second:

$$(y + 4) + y^2 - 24 = 0 \iff y^2 + y - 20 = 0 \iff (y - 4)(y + 5) = 0,$$

so $y = 4$ or $y = -5$.

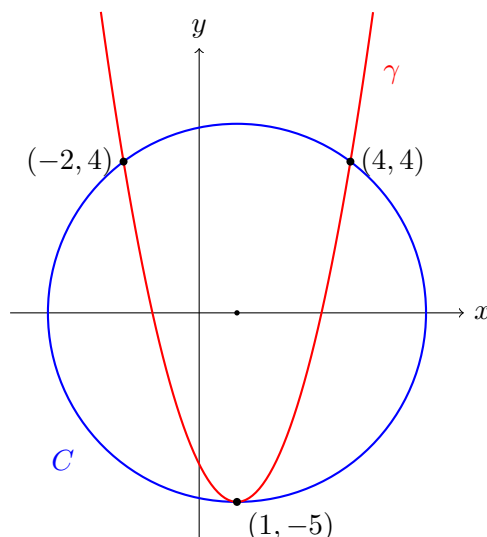
- $y = 4$: $x^2 - 2x - 4 = 4 \iff x^2 - 2x - 8 = 0 \iff (x - 4)(x + 2) = 0$, so $x = 4$ or $x = -2$.
- $y = -5$: $x^2 - 2x - 4 = -5 \iff x^2 - 2x + 1 = 0 \iff (x - 1)^2 = 0$, so $x = 1$ (double root).

Hence

$$\gamma \cap C = \{(-2, 4), (4, 4), (1, -5)\}.$$

Check, for instance, $(4, 4)$: $16 + 16 - 8 - 24 = 0$.

Drawing. $C: (x - 1)^2 + y^2 = 25$ is the circle with center $(1, 0)$ and radius 5. The parabola has vertex $x_V = 1$, $y_V = 1 - 2 - 4 = -5$, that is $V = (1, -5)$: the vertex is the lowest point of the circle. The double root $x = 1$ means that at V the parabola and the circle are tangent (both have horizontal tangent line $y = -5$). The parabola meets the x -axis where $x^2 - 2x - 4 = 0$, that is $x = 1 \pm \sqrt{5}$, and the y -axis at $(0, -4)$.



Exercise III. Find the parabola γ (with vertical axis) such that $(1, 2) \in \gamma$, $(0, 3) \in \gamma$, $(3, 5) \in \gamma$.

Solution. We look for $\gamma: y = ax^2 + bx + c$ with $a \neq 0$. Each point gives a linear equation in a, b, c :

$$\begin{cases} (0, 3): & c = 3 \\ (1, 2): & a + b + c = 2 \\ (3, 5): & 9a + 3b + c = 5. \end{cases}$$

Substituting $c = 3$: $a + b = -1$ and $9a + 3b = 2$. From the first, $b = -1 - a$; then $9a - 3 - 3a = 2$, so $6a = 5$, $a = \frac{5}{6}$ and $b = -1 - \frac{5}{6} = -\frac{11}{6}$. Since $a \neq 0$ this is indeed a parabola:

$$\gamma: y = \frac{5}{6}x^2 - \frac{11}{6}x + 3, \quad \text{equivalently} \quad 6y = 5x^2 - 11x + 18.$$

Check: $x = 1$: $\frac{5-11+18}{6} = 2$; $x = 0$: 3 ; $x = 3$: $\frac{45-33+18}{6} = 5$.

Exercise IV. Find the circle C such that $(1, 2) \in C$, $(0, 3) \in C$, $(3, 5) \in C$.

Solution. The three points are not collinear (the slope from $(0, 3)$ to $(1, 2)$ is -1 , the slope from $(0, 3)$ to $(3, 5)$ is $\frac{2}{3}$), so there is exactly one circle through them. We look for $C: x^2 + y^2 + Dx + Ey + F = 0$:

$$\begin{cases} (1, 2): & 1 + 4 + D + 2E + F = 0 \\ (0, 3): & 9 + 3E + F = 0 \\ (3, 5): & 9 + 25 + 3D + 5E + F = 0 \end{cases} \iff \begin{cases} D + 2E + F = -5 \\ 3E + F = -9 \\ 3D + 5E + F = -34. \end{cases}$$

From the second, $F = -9 - 3E$. Substituting:

$$D + 2E - 9 - 3E = -5 \iff D - E = 4, \quad 3D + 5E - 9 - 3E = -34 \iff 3D + 2E = -25.$$

From $D = E + 4$: $3E + 12 + 2E = -25$, so $5E = -37$, $E = -\frac{37}{5}$, $D = -\frac{17}{5}$, and $F = -9 + \frac{111}{5} = \frac{66}{5}$. Multiplying by 5:

$$C: 5x^2 + 5y^2 - 17x - 37y + 66 = 0.$$

Center and radius. Center $\left(-\frac{D}{2}, -\frac{E}{2}\right) = \left(\frac{17}{10}, \frac{37}{10}\right)$ and

$$r^2 = \frac{D^2 + E^2}{4} - F = \frac{289 + 1369}{100} - \frac{1320}{100} = \frac{338}{100}, \quad r = \frac{\sqrt{338}}{10} = \frac{13\sqrt{2}}{10}.$$

Check with $(0, 3)$: $\left(\frac{17}{10}\right)^2 + \left(3 - \frac{37}{10}\right)^2 = \frac{289+49}{100} = \frac{338}{100} = r^2$.

Exercise V. Let $\gamma: y = x^2 - 3x$ and $C: x^2 + y^2 + 2x - 15 = 0$. Find $\gamma \cap C$.

Solution. *The curves.* Completing the square, $C: (x+1)^2 + y^2 = 16$: circle with center $(-1, 0)$ and radius 4. The parabola is the same as in Exercise I: vertex $\left(\frac{3}{2}, -\frac{9}{4}\right)$, zeros $x = 0$ and $x = 3$.

The system.

$$\begin{cases} y = x^2 - 3x \\ x^2 + y^2 + 2x - 15 = 0. \end{cases}$$

Substituting the first equation into the second:

$$\begin{aligned} x^2 + (x^2 - 3x)^2 + 2x - 15 &= 0 \iff x^4 - 6x^3 + 9x^2 + x^2 + 2x - 15 = 0 \\ &\iff p(x) := x^4 - 6x^3 + 10x^2 + 2x - 15 = 0. \end{aligned}$$

Rational roots. The candidates are $\pm 1, \pm 3, \pm 5, \pm 15$. We find

$$p(-1) = 1 + 6 + 10 - 2 - 15 = 0, \quad p(3) = 81 - 162 + 90 + 6 - 15 = 0.$$

So $p(x)$ is divisible by $(x+1)(x-3) = x^2 - 2x - 3$. Long division gives

$$x^4 - 6x^3 + 10x^2 + 2x - 15 = (x^2 - 2x - 3)(x^2 - 4x + 5).$$

(Check: $(x^2 - 2x - 3)(x^2 - 4x + 5) = x^4 - 4x^3 + 5x^2 - 2x^3 + 8x^2 - 10x - 3x^2 + 12x - 15 = x^4 - 6x^3 + 10x^2 + 2x - 15$.)

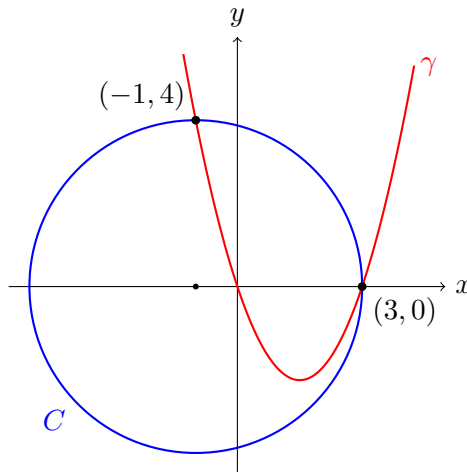
The factor $x^2 - 4x + 5$ has discriminant $\Delta = 16 - 20 = -4 < 0$: no real roots. Hence the real solutions are $x = -1$ and $x = 3$ only.

The points. From $y = x^2 - 3x$:

$$x = -1: y = 1 + 3 = 4, \quad x = 3: y = 9 - 9 = 0.$$

$$\gamma \cap C = \{(-1, 4), (3, 0)\}.$$

Check: $(-1, 4)$: $1 + 16 - 2 - 15 = 0$; $(3, 0)$: $9 + 0 + 6 - 15 = 0$. Note that $(-1, 4)$ is the highest point of C (center $(-1, 0)$, radius 4) and $(3, 0)$ is its rightmost point.



Exercise VI. Find, if possible, center and radius of $x^2 + y^2 - 2x - 2y + 7 = 0$.

Solution. Completing the squares:

$$(x^2 - 2x + 1) + (y^2 - 2y + 1) - 1 - 1 + 7 = 0 \iff (x-1)^2 + (y-1)^2 = -5.$$

The left-hand side is ≥ 0 for every (x, y) , while the right-hand side is negative: no point of the plane satisfies the equation. Equivalently, $\frac{D^2+E^2}{4} - F = \frac{4+4}{4} - 7 = -5 < 0$. So the equation does *not* represent a circle (its set of points is empty), and center and radius do not exist.

Exercise VII. Find the line r such that $r \perp \{x + y + 1 = 0\}$ and $(1, 1) \in r$.

Solution. The line $x + y + 1 = 0$ is $y = -x - 1$, with slope $m = -1$. A line perpendicular to it has slope m' with $m m' = -1$, so $m' = 1$. The line through $(1, 1)$ with slope 1 is $y - 1 = 1 \cdot (x - 1)$, that is

$$r: y = x \quad (\text{equivalently } x - y = 0).$$

Check: the direction vectors $(1, -1)$ of the given line and $(1, 1)$ of r have scalar product $1 - 1 = 0$, and $(1, 1)$ satisfies $y = x$.

Exercise VIII. Find the distance between $(3, -1)$ and $(2, 4)$.

Solution.

$$d = \sqrt{(2 - 3)^2 + (4 - (-1))^2} = \sqrt{1 + 25} = \sqrt{26} \approx 5.10.$$