

Linear Systems — Solutions

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Exercise 1

Exercise. Solve the system

$$\begin{cases} 4x + y - 3z = 5 & (E_1) \\ 2x + y - 5z = 0 & (E_2) \\ 7x + 2y - 6z = 3 & (E_3) \end{cases}$$

Solution. The variable y has the simplest coefficients, so we eliminate it first.

Step 1: eliminate y from E_1 . Compute $E_1 - E_2$:

$$(4x + y - 3z) - (2x + y - 5z) = 5 - 0 \implies 2x + 2z = 5. \quad (E_4)$$

Step 2: eliminate y from E_3 . Compute $E_3 - 2E_2$:

$$(7x + 2y - 6z) - (4x + 2y - 10z) = 3 - 0 \implies 3x + 4z = 3. \quad (E_5)$$

Step 3: solve the 2×2 system in x, z .

$$\begin{cases} 2x + 2z = 5 \\ 3x + 4z = 3 \end{cases}$$

From E_4 : $x = \frac{5}{2} - z$. Substituting in E_5 :

$$3\left(\frac{5}{2} - z\right) + 4z = 3 \implies \frac{15}{2} + z = 3 \implies z = 3 - \frac{15}{2} = -\frac{9}{2}.$$

Then

$$x = \frac{5}{2} - \left(-\frac{9}{2}\right) = \frac{14}{2} = 7.$$

Step 4: back-substitute in E_2 to find y . From E_2 , $y = -2x + 5z$, so

$$y = -2 \cdot 7 + 5\left(-\frac{9}{2}\right) = -14 - \frac{45}{2} = -\frac{73}{2}.$$

Step 5: check in all three equations.

$$E_1: \quad 4 \cdot 7 - \frac{73}{2} - 3\left(-\frac{9}{2}\right) = 28 - \frac{73}{2} + \frac{27}{2} = 28 - 23 = 5 \quad \checkmark$$

$$E_2: \quad 2 \cdot 7 - \frac{73}{2} - 5\left(-\frac{9}{2}\right) = 14 - \frac{73}{2} + \frac{45}{2} = 14 - 14 = 0 \quad \checkmark$$

$$E_3: \quad 7 \cdot 7 + 2\left(-\frac{73}{2}\right) - 6\left(-\frac{9}{2}\right) = 49 - 73 + 27 = 3 \quad \checkmark$$

Conclusion. The system has exactly one solution:

$$(x, y, z) = \left(7, -\frac{73}{2}, -\frac{9}{2}\right).$$

□

Remark. Uniqueness can also be seen from the coefficient matrix:

$$\det \begin{pmatrix} 4 & 1 & -3 \\ 2 & 1 & -5 \\ 7 & 2 & -6 \end{pmatrix} = 4(-6 + 10) - 1(-12 + 35) + (-3)(4 - 7) = 16 - 23 + 9 = 2 \neq 0.$$

Exercise 2

Exercise. Solve the system

$$\begin{cases} x + y + z = 0 & (E_1) \\ x - y + 2z = 0 & (E_2) \\ x - y - y = 1 & (E_3) \end{cases}$$

Solution. First simplify E_3 : $x - y - y = x - 2y$, so the system is

$$\begin{cases} x + y + z = 0 \\ x - y + 2z = 0 \\ x - 2y = 1 \end{cases}$$

Step 1: eliminate x between E_1 and E_2 . Compute $E_1 - E_2$:

$$(x + y + z) - (x - y + 2z) = 0 - 0 \implies 2y - z = 0 \implies z = 2y.$$

Step 2: express x in terms of y . From E_1 :

$$x = -y - z = -y - 2y = -3y.$$

Step 3: substitute in E_3 .

$$x - 2y = -3y - 2y = -5y = 1 \implies y = -\frac{1}{5}.$$

Hence

$$x = -3\left(-\frac{1}{5}\right) = \frac{3}{5}, \quad z = 2\left(-\frac{1}{5}\right) = -\frac{2}{5}.$$

Step 4: check in all three equations.

$$\begin{aligned} E_1: \quad & \frac{3}{5} - \frac{1}{5} - \frac{2}{5} = 0 \quad \checkmark \\ E_2: \quad & \frac{3}{5} + \frac{1}{5} - \frac{4}{5} = 0 \quad \checkmark \\ E_3: \quad & \frac{3}{5} + \frac{1}{5} + \frac{1}{5} = 1 \quad \checkmark \end{aligned}$$

Conclusion. The system has exactly one solution:

$$(x, y, z) = \left(\frac{3}{5}, -\frac{1}{5}, -\frac{2}{5}\right).$$

□

Remark. If the third equation is read as $x - y - z = 1$, the same method gives: $E_1 - E_2$ still yields $z = 2y$; $E_1 - E_3$ yields $2y + 2z = -1$, i.e. $6y = -1$, so $y = -\frac{1}{6}$, $z = -\frac{1}{3}$ and $x = -y - z = \frac{1}{2}$. Check: $\frac{1}{2} - \frac{1}{6} - \frac{1}{3} = 0$, $\frac{1}{2} + \frac{1}{6} - \frac{2}{3} = 0$, $\frac{1}{2} + \frac{1}{6} + \frac{1}{3} = 1$. The solution is then

$$(x, y, z) = \left(\frac{1}{2}, -\frac{1}{6}, -\frac{1}{3}\right).$$