Parabolas through three points and intersections

Caboara

1 Finding a parabola through three points

We look for a parabola with vertical axis, i.e. the graph of

$$y = ax^2 + bx + c, \qquad a \neq 0.$$

Each point (x_i, y_i) on the parabola gives one linear equation in the unknowns a, b, c:

$$ax_i^2 + bx_i + c = y_i.$$

Three points give a linear system of three equations in three unknowns. At the end we must check that $a \neq 0$: if $a = 0$ the curve is a line, not a parabola.

Exercise 1. Find the parabola $y = ax^2 + bx + c$ passing through the points $A = (0, 1)$, $B = (1, 2)$, $C = (2, 7)$.

Solution. We substitute the coordinates of each point into $y = ax^2 + bx + c$:

$$\begin{aligned}
A = (0,1): &\quad a \cdot 0^2 + b \cdot 0 + c = 1 &&\implies c = 1\\
B = (1,2): &\quad a \cdot 1^2 + b \cdot 1 + c = 2 &&\implies a + b + c = 2\\
C = (2,7): &\quad a \cdot 2^2 + b \cdot 2 + c = 7 &&\implies 4a + 2b + c = 7
\end{aligned}$$

The first equation gives $c = 1$ immediately. Substituting $c = 1$ into the other two:

$$\begin{cases} a + b = 1 \\ 4a + 2b = 6 \end{cases} \implies \begin{cases} a + b = 1 \\ 2a + b = 3 \end{cases}$$

(we divided the second equation by 2). Subtracting the first equation from the second:

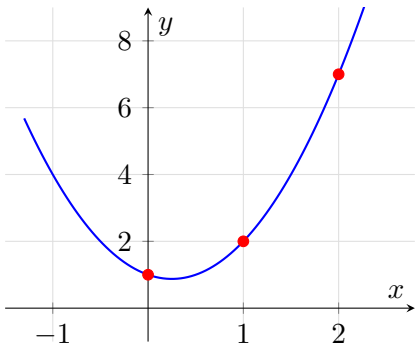
$$(2a + b) - (a + b) = 3 - 1 \quad \implies \quad a = 2.$$

Then $b = 1 - a = 1 - 2 = -1$.

Since $a = 2 \neq 0$, we have a parabola:

$$\boxed{y = 2x^2 - x + 1}$$

Check. $x = 0$: $y = 1$; $\quad x = 1$: $y = 2 - 1 + 1 = 2$; $\quad x = 2$: $y = 8 - 2 + 1 = 7$. All three points lie on the parabola.



Exercise 2. Find the parabola $y = ax^2 + bx + c$ passing through the points $A = (-1, 6)$, $B = (1, 0)$, $C = (2, 0)$.

Solution. Substituting the three points:

$$\begin{aligned} A = (-1, 6): &\quad a(-1)^2 + b(-1) + c = 6 \qquad\qquad \Longrightarrow \quad a - b + c = 6 \\ B = (1, 0): &\quad a + b + c = 0 \\ C = (2, 0): &\quad 4a + 2b + c = 0 \end{aligned}$$

We eliminate c by subtracting equations.

Second minus first:

$$(a + b + c) - (a - b + c) = 0 - 6 \quad \Longrightarrow \quad 2b = -6 \quad \Longrightarrow \quad b = -3.$$

Third minus second:

$$(4a + 2b + c) - (a + b + c) = 0 - 0 \quad \Longrightarrow \quad 3a + b = 0.$$

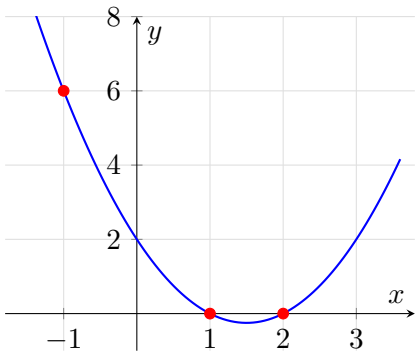
With $b = -3$ we get $3a - 3 = 0$, so $a = 1$. Finally, from the second equation, $c = -a - b = -1 + 3 = 2$.

Since $a = 1 \neq 0$, we have a parabola:

$$\boxed{y = x^2 - 3x + 2}$$

Check. $x = -1$: $y = 1 + 3 + 2 = 6$; $\quad x = 1$: $y = 1 - 3 + 2 = 0$; $\quad x = 2$: $y = 4 - 6 + 2 = 0$.

Remark. The points B and C lie on the x-axis, so 1 and 2 are the roots of the parabola, and indeed $x^2 - 3x + 2 = (x - 1)(x - 2)$. We could have started from $y = a(x - 1)(x - 2)$ and found a from the point A: $6 = a(-2)(-3) = 6a$, so $a = 1$.



Exercise 3. Find the parabola $y = ax^2 + bx + c$ passing through the points $A = (1, 3)$, $B = (2, 5)$, $C = (4, 9)$.

Solution. Substituting the three points:

$$\begin{aligned} A = (1, 3): &\quad a + b + c = 3 \\ B = (2, 5): &\quad 4a + 2b + c = 5 \\ C = (4, 9): &\quad 16a + 4b + c = 9 \end{aligned}$$

Second minus first:

$$3a + b = 2.$$

Third minus second:

$$12a + 2b = 4 \quad \Longrightarrow \quad 6a + b = 2.$$

Subtracting these two equations:

$$(6a+b)-(3a+b)=2-2 \quad\Longrightarrow\quad 3a=0 \quad\Longrightarrow\quad a=0.$$

Then $b=2-3a=2$ and $c=3-a-b=1$.

The only solution of the system is $a=0$, $b=2$, $c=1$, that is, $y=2x+1$. This is a *line*, not a parabola.

Conclusion. There is **no parabola** through A, B, C.

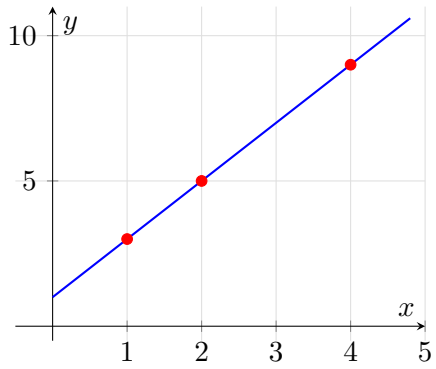
Why. The three points are collinear. The slope of AB is

$$m_{AB}=\frac{5-3}{2-1}=2,$$

and the slope of BC is

$$m_{BC}=\frac{9-5}{4-2}=2.$$

The slopes are equal, so A, B, C lie on the same line $y=2x+1$. A parabola and a line meet in at most two points: substituting $y=mx+q$ into $y=ax^2+bx+c$ gives an equation of degree 2 in x, which has at most two solutions. Hence no parabola can pass through three collinear points.



2 Intersection of two lines

To find the intersection point of two lines, we solve the system formed by their equations. If the lines are not parallel, there is exactly one solution (x,y): the intersection point.

Exercise 4. Find the intersection point of the lines

$$r:\ 2x-y=1, \qquad s:\ x+y=5.$$

Solution. We solve the system

$$\begin{cases} 2x-y=1 \\ x+y=5 \end{cases}$$

Adding the two equations, y cancels:

$$(2x-y)+(x+y)=1+5 \quad\Longrightarrow\quad 3x=6 \quad\Longrightarrow\quad x=2.$$

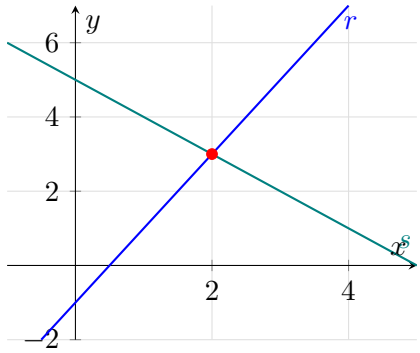
Substituting $x=2$ into the second equation: $2+y=5$, so $y=3$.

The intersection point is

$$\boxed{P=(2,3)}$$

Check. r: $2\cdot 2-3=1$; $\quad s$: $2+3=5$.

Alternative method. In explicit form the lines are $y=2x-1$ and $y=-x+5$. Setting the two expressions for y equal: $2x-1=-x+5$, so $3x=6$ and $x=2$, $y=2\cdot 2-1=3$.



Exercise 5. Find the intersection point of the lines

$$r: \ 3x - 2y = 4, \qquad s: \ y = 4.$$

Solution. The line s is horizontal: every point of s has $y = 4$. So the intersection point also has $y = 4$, and we only need to find x.

Substituting $y = 4$ into the equation of r:

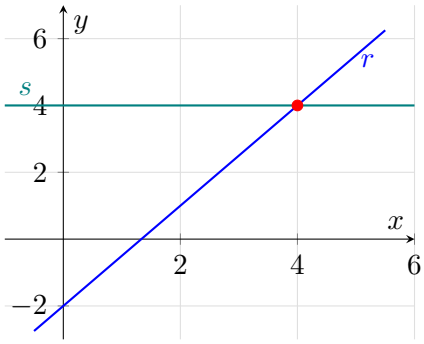
$$3x - 2 \cdot 4 = 4 \quad \Longrightarrow \quad 3x - 8 = 4 \quad \Longrightarrow \quad 3x = 12 \quad \Longrightarrow \quad x = 4.$$

The intersection point is

$$\boxed{P = (4, 4)}$$

Check. r: $3 \cdot 4 - 2 \cdot 4 = 12 - 8 = 4$; $\quad s$: $y = 4$.

Remark. A horizontal line $y = k$ meets every non-horizontal line in exactly one point. It does not meet another horizontal line $y = k'$ with $k' \neq k$ (the lines are parallel).



3 Intersection of a line and a parabola

To find the intersection points of the parabola $y = ax^2 + bx + c$ and the line $y = mx + q$, we set the two expressions for y equal:

$$ax^2 + bx + c = mx + q \quad \Longleftrightarrow \quad ax^2 + (b - m)x + (c - q) = 0.$$

This is an equation of degree 2. Its discriminant Δ tells us the number of intersection points:

- $\Delta > 0$: two distinct intersection points (the line is *secant*);
- $\Delta = 0$: one intersection point (the line is *tangent*);
- $\Delta < 0$: no intersection points (the line is *external*).

For each solution x, we find y from the equation of the line (it is simpler than the parabola).

Exercise 6. Find the intersection points of the parabola $y = x^2 - 2x - 3$ and the line $y = x + 1$.

Solution. We set the two expressions for y equal:

$$x^2 - 2x - 3 = x + 1 \quad \Longrightarrow \quad x^2 - 3x - 4 = 0.$$

The discriminant is

$$\Delta = (-3)^2 - 4 \cdot 1 \cdot (-4) = 9 + 16 = 25 > 0,$$

so there are two intersection points. The solutions are

$$x = \frac{3 \pm \sqrt{25}}{2} = \frac{3 \pm 5}{2}, \qquad x_1 = 4, \quad x_2 = -1.$$

(Equivalently, $x^2 - 3x - 4 = (x - 4)(x + 1)$.)

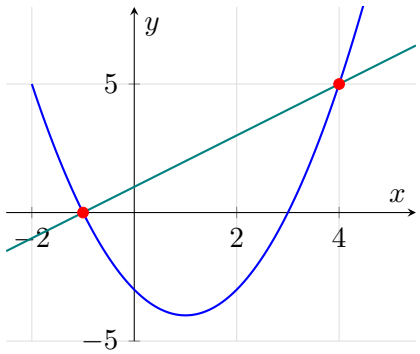
We find y from the line $y = x + 1$:

$$x_1 = 4 \implies y_1 = 5, \qquad x_2 = -1 \implies y_2 = 0.$$

The intersection points are

$$\boxed{P_1 = (4, 5), \qquad P_2 = (-1, 0)}$$

Check on the parabola. $x = 4$: $16 - 8 - 3 = 5$; $\quad x = -1$: $1 + 2 - 3 = 0$.



Exercise 7. Find the intersection points of the parabola $y = x^2 + 1$ and the line $y = 2x$.

Solution. We set the two expressions for y equal:

$$x^2 + 1 = 2x \quad \Longrightarrow \quad x^2 - 2x + 1 = 0.$$

The discriminant is

$$\Delta = (-2)^2 - 4 \cdot 1 \cdot 1 = 4 - 4 = 0,$$

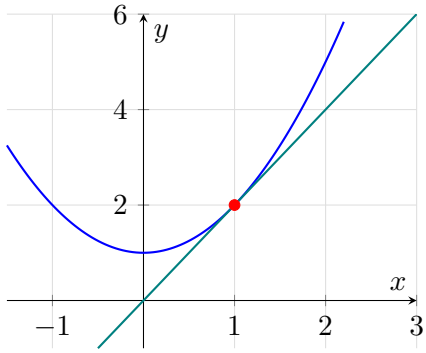
so there is exactly one intersection point. Indeed $x^2 - 2x + 1 = (x - 1)^2$, and the only solution is $x = 1$.

From the line, $y = 2 \cdot 1 = 2$. The intersection point is

$$\boxed{P = (1, 2)}$$

Check on the parabola. $x = 1$: $1 + 1 = 2$.

The line $y = 2x$ is *tangent* to the parabola at P.



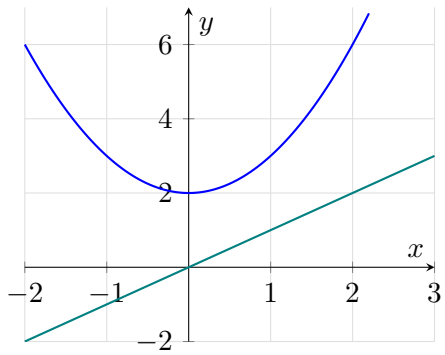
Exercise 8. Find the intersection points of the parabola $y = x^2 + 2$ and the line $y = x$.

Solution. We set the two expressions for y equal:

$$x^2 + 2 = x \quad \Longrightarrow \quad x^2 - x + 2 = 0.$$

The discriminant is

$$\Delta = (-1)^2 - 4 \cdot 1 \cdot 2 = 1 - 8 = -7 < 0.$$

The equation has no real solutions, so the line and the parabola have **no intersection points**: the line is external to the parabola.

Remark. We can also see this directly: $x^2 - x + 2 = \left(x - \frac{1}{2}\right)^2 + \frac{7}{4} \geq \frac{7}{4} > 0$ for every x. So the parabola always lies above the line.