# Analytic Geometry: Points, Lines, Parabolas and Circles

Exercise sheet 2 – with step-by-step solutions

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## 1 Distance between two points

**Reminder.** The distance between $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

**Exercise 1.1.** Compute the distance between $A(-2, 1)$ and $B(4, 9)$.

*Solution. Step 1: differences of coordinates.* $x_2 - x_1 = 4 - (-2) = 6, \quad y_2 - y_1 = 9 - 1 = 8.$
*Step 2: squares and sum.* $6^2 + 8^2 = 36 + 64 = 100.$
*Step 3: square root.* $d(A, B) = \sqrt{100} = 10.$ □

**Exercise 1.2.** Compute the distance between $P(3, -4)$ and $Q(-1, 2)$.

*Solution. Step 1.* $x_Q - x_P = -1 - 3 = -4, \quad y_Q - y_P = 2 - (-4) = 6.$
*Step 2.* $(-4)^2 + 6^2 = 16 + 36 = 52.$
*Step 3.* $d(P, Q) = \sqrt{52} = \sqrt{4 \cdot 13} = 2\sqrt{13} \approx 7.21.$ □

## 2 Circle through three points

**Method.** Write the circle in the general form

$$x^2 + y^2 + ax + by + c = 0.$$

Substituting the coordinates of the three points gives a linear system in $a, b, c$. If the system has a (unique) solution, the circle has

$$\text{center } C\left(-\tfrac{a}{2}, -\tfrac{b}{2}\right), \qquad r = \sqrt{\tfrac{a^2}{4} + \tfrac{b^2}{4} - c}.$$

If the three points are collinear the system is impossible: no circle passes through them.

**Exercise 2.1.** Find the circle through $A(1, 5)$, $B(3, 1)$, $D(-6, -2)$.

*Solution. Step 1: substitute the points.*

$$\begin{aligned} A(1,5): &\quad 1 + 25 + a + 5b + c = 0 &&\Longrightarrow \quad a + 5b + c = -26 \\ B(3,1): &\quad 9 + 1 + 3a + b + c = 0 &&\Longrightarrow \quad 3a + b + c = -10 \\ D(-6,-2): &\quad 36 + 4 - 6a - 2b + c = 0 &&\Longrightarrow \quad -6a - 2b + c = -40 \end{aligned}$$

*Step 2: eliminate $c$.*

$$\begin{aligned} (1\text{st}) - (2\text{nd}): &\quad -2a + 4b = -16 &&\Longrightarrow \quad a = 2b + 8, \\ (2\text{nd}) - (3\text{rd}): &\quad 9a + 3b = 30 &&\Longrightarrow \quad 3a + b = 10. \end{aligned}$$

*Step 3: solve.* $3(2b + 8) + b = 10 \Rightarrow 7b = -14 \Rightarrow b = -2$, hence $a = 4$. From the second equation: $c = -10 - 3a - b = -10 - 12 + 2 = -20$.

*Step 4: equation, center, radius.*

$$x^2+y^2+4x-2y-20=0, \qquad C(-2,1), \qquad r=\sqrt{4+1+20}=\sqrt{25}=5.$$

*Check:* $d(C,A)=\sqrt{3^2+4^2}=5$, $d(C,B)=\sqrt{5^2+0^2}=5$, $d(C,D)=\sqrt{(-4)^2+(-3)^2}=5$. □

**Exercise 2.2.** Find the circle through $A(5,2)$, $B(0,1)$, $D(1,-4)$.

*Solution. Step 1: substitute the points.*

$$\begin{aligned} A(5,2): &\quad 25+4+5a+2b+c=0 &\Longrightarrow &\quad 5a+2b+c=-29\\ B(0,1): &\quad 0+1+0+b+c=0 &\Longrightarrow &\quad b+c=-1\\ D(1,-4): &\quad 1+16+a-4b+c=0 &\Longrightarrow &\quad a-4b+c=-17 \end{aligned}$$

*Step 2: eliminate* $c$ by subtracting the second equation from the other two:

$$\begin{aligned} &5a+b=-28,\\ &a-5b=-16 &\Longrightarrow &\quad a=5b-16. \end{aligned}$$

*Step 3: solve.* $5(5b-16)+b=-28 \Rightarrow 26b=52 \Rightarrow b=2$, so $a=-6$. Then $c=-1-b=-3$.
*Step 4: equation, center, radius.*

$$x^2+y^2-6x+2y-3=0, \qquad C(3,-1), \qquad r=\sqrt{9+1+3}=\sqrt{13}.$$

*Check:* $d(C,A)^2=2^2+3^2=13$, $d(C,B)^2=(-3)^2+2^2=13$, $d(C,D)^2=(-2)^2+(-3)^2=13$.
□

**Exercise 2.3.** Find, if it exists, the circle through $A(0,2)$, $B(1,5)$, $D(-2,-4)$.

*Solution. Step 1: substitute the points.*

$$\begin{aligned} A(0,2): &\quad 4+2b+c=0 &\Longrightarrow &\quad 2b+c=-4\\ B(1,5): &\quad 26+a+5b+c=0 &\Longrightarrow &\quad a+5b+c=-26\\ D(-2,-4): &\quad 20-2a-4b+c=0 &\Longrightarrow &\quad -2a-4b+c=-20 \end{aligned}$$

*Step 2: eliminate* $c$ by subtracting the first equation from the other two:

$$\begin{aligned} &a+3b=-22,\\ &-2a-6b=-16 &\Longrightarrow &\quad a+3b=8. \end{aligned}$$

*Step 3: conclusion.* The quantity $a+3b$ cannot be both $-22$ and 8: the system is **impossible**, so **no circle** passes through the three points.

*Geometric reason.* The slopes are $m_{AB}=\frac{5-2}{1-0}=3$ and $m_{AD}=\frac{-4-2}{-2-0}=3$: the points are collinear (they lie on the line $y=3x+2$), and three distinct collinear points never lie on a circle.
□

## 3 Circle with given center and radius

**Reminder.** The circle with center $C(x_0,y_0)$ and radius $r>0$ is

$$(x-x_0)^2+(y-y_0)^2=r^2.$$

**Exercise 3.1.** Write the equation of the circle with center $C(-4,1)$ and radius $r=3$.

*Solution. Step 1: standard form.* $(x+4)^2+(y-1)^2=9$.
*Step 2: expand.* $x^2+8x+16+y^2-2y+1=9$.

*Step 3: general form.* $x^2+y^2+8x-2y+8=0$.
*Check:* $a=8$, $b=-2$, $c=8$ give $C(-4,1)$ and $r=\sqrt{16+1-8}=3$. □

**Exercise 3.2.** Write the equation of the circle with center $C\left(\frac{2}{3},-1\right)$ and radius $r=2$.

*Solution. Step 1: standard form.* $\left(x-\frac{2}{3}\right)^2+(y+1)^2=4$.
*Step 2: expand.* $x^2-\frac{4}{3}x+\frac{4}{9}+y^2+2y+1=4$.
*Step 3: general form.* Since $\frac{4}{9}+1-4=\frac{4-27}{9}=-\frac{23}{9}$,

$$x^2+y^2-\tfrac{4}{3}x+2y-\tfrac{23}{9}=0 \quad \text{or, multiplying by 9,} \quad 9x^2+9y^2-12x+18y-23=0.$$

*Check:* $a=-\frac{4}{3}$, $b=2$, $c=-\frac{23}{9}$ give $C\left(\frac{2}{3},-1\right)$ and $r^2=\frac{4}{9}+1+\frac{23}{9}=\frac{36}{9}=4$. □

## 4 Center and radius from the equation

**Method.** An equation $\alpha x^2+\alpha y^2+\cdots=0$ (same coefficient of $x^2$ and $y^2$, no $xy$ term) is first divided by $\alpha$ to get $x^2+y^2+ax+by+c=0$. Completing the squares,

$$\left(x+\tfrac{a}{2}\right)^2+\left(y+\tfrac{b}{2}\right)^2=\tfrac{a^2}{4}+\tfrac{b^2}{4}-c.$$

It is a circle only if the right-hand side is *positive*.

**Exercise 4.1.** Find center and radius of $x^2+y^2+6x-4y-12=0$.

*Solution. Step 1: group.* $(x^2+6x)+(y^2-4y)-12=0$.
*Step 2: complete the squares.* $x^2+6x=(x+3)^2-9$, $\quad y^2-4y=(y-2)^2-4$, so

$$(x+3)^2-9+(y-2)^2-4-12=0 \implies (x+3)^2+(y-2)^2=25.$$

*Step 3: read off.* Since $25>0$: center $C(-3,2)$, radius $r=5$.
*With the formulas:* $a=6$, $b=-4$, $c=-12$: $C(-3,2)$, $r^2=9+4+12=25$. □

**Exercise 4.2.** Find center and radius of $4x^2+4y^2-4x+12y-6=0$.

*Solution. Step 1: normalize.* The coefficients of $x^2$ and $y^2$ are equal; divide by 4:

$$x^2+y^2-x+3y-\tfrac{3}{2}=0.$$

*Step 2: complete the squares.* $x^2-x=\left(x-\frac{1}{2}\right)^2-\frac{1}{4}$, $\quad y^2+3y=\left(y+\frac{3}{2}\right)^2-\frac{9}{4}$, so

$$\left(x-\tfrac{1}{2}\right)^2+\left(y+\tfrac{3}{2}\right)^2=\tfrac{1}{4}+\tfrac{9}{4}+\tfrac{3}{2}=\tfrac{16}{4}=4.$$

*Step 3: read off.* Center $C\left(\frac{1}{2},-\frac{3}{2}\right)$, radius $r=2$.
*Warning:* applying the formulas to the *non*-divided equation would give a wrong result. □

**Exercise 4.3.** Find, if possible, center and radius of $x^2+y^2+4x-6y+15=0$.

*Solution. Step 1: complete the squares.* $x^2+4x=(x+2)^2-4$, $\quad y^2-6y=(y-3)^2-9$, so

$$(x+2)^2-4+(y-3)^2-9+15=0 \implies (x+2)^2+(y-3)^2=-2.$$

*Step 2: conclusion.* The left-hand side is a sum of squares, hence $\geq 0$ for every real $(x,y)$, while the right-hand side is $-2<0$. No real point satisfies the equation: it is **not a circle**.
*With the formula:* $r^2=\frac{a^2}{4}+\frac{b^2}{4}-c=4+9-15=-2<0$.
*Remark.* With constant term 13 instead of 15 we would get $(x+2)^2+(y-3)^2=0$, satisfied only by the point $(-2,3)$: a "circle of radius 0", again not a genuine circle. □

## 5 Intersections of a line and a circle

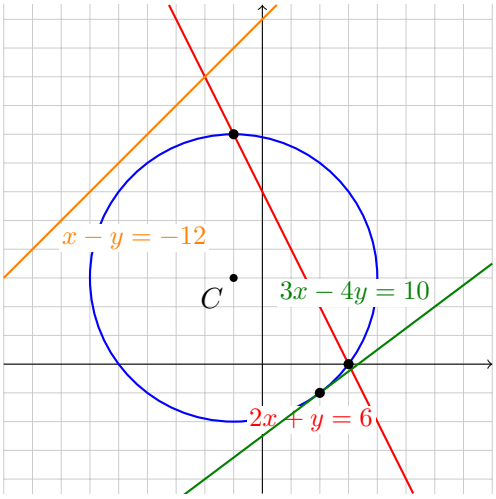
**Method.** Solve the system circle–line by substitution: one obtains a quadratic equation, whose discriminant $\Delta$ tells us the number of intersection points:

$$\Delta > 0:\ 2 \text{ points (secant)}, \qquad \Delta = 0:\ 1 \text{ point (tangent)}, \qquad \Delta < 0:\ \text{no points (external line)}.$$

Equivalently, compare the distance $d$ of the center from the line with the radius $r$: $d < r$, $d = r$, $d > r$ respectively. Recall $d\big(P_0,\ \alpha x + \beta y + \gamma = 0\big) = \dfrac{|\alpha x_0 + \beta y_0 + \gamma|}{\sqrt{\alpha^2 + \beta^2}}$.

In all three exercises the circle is

$$\Gamma:\ x^2 + y^2 + 2x - 6y - 15 = 0, \qquad \text{center } C(-1, 3), \quad r = \sqrt{1 + 9 + 15} = 5.$$




**Exercise 5.1.** Find the intersections of $\Gamma$ with the line $s:\ 2x + y - 6 = 0$.

*Solution. Step 1: solve the line for $y$.* $y = 6 - 2x$.

*Step 2: substitute into* $\Gamma$.

$$\begin{aligned} & x^2 + (6 - 2x)^2 + 2x - 6(6 - 2x) - 15 = 0 \\ \Longrightarrow\quad & x^2 + 36 - 24x + 4x^2 + 2x - 36 + 12x - 15 = 0 \\ \Longrightarrow\quad & 5x^2 - 10x - 15 = 0. \end{aligned}$$

*Step 3: solve.* Dividing by 5: $x^2 - 2x - 3 = 0$, with $\Delta = 4 + 12 = 16 > 0$:

$$x = \frac{2 \pm 4}{2} \quad \Longrightarrow \quad x_1 = 3,\ x_2 = -1.$$

*Step 4: find $y$.* $y_1 = 6 - 6 = 0$, $y_2 = 6 + 2 = 8$. **Two intersection points:** $(3, 0)$ and $(-1, 8)$; the line is secant.

*Check with the distance:* $d(C, s) = \dfrac{|-2 + 3 - 6|}{\sqrt{4 + 1}} = \dfrac{5}{\sqrt{5}} = \sqrt{5} \approx 2.24 < 5.$ □

**Exercise 5.2.** Find the intersections of $\Gamma$ with the line $t:\ 3x - 4y - 10 = 0$.

*Solution. Step 1: solve the line for $y$.* $y = \dfrac{3x - 10}{4}$.

*Step 2: substitute into* $\Gamma$ and multiply everything by 16 to clear denominators:

$$\begin{aligned} & 16x^2 + (3x - 10)^2 + 32x - 24(3x - 10) - 240 = 0 \\ \Longrightarrow\quad & 16x^2 + 9x^2 - 60x + 100 + 32x - 72x + 240 - 240 = 0 \\ \Longrightarrow\quad & 25x^2 - 100x + 100 = 0. \end{aligned}$$

*Step 3: solve.* Dividing by 25: $x^2 - 4x + 4 = 0$, i.e. $(x-2)^2 = 0$, so $\Delta = 0$ and $x = 2$ (double root).

*Step 4: find $y$.* $y = \dfrac{6-10}{4} = -1$. **One intersection point:** $(2, -1)$; the line is tangent to $\Gamma$.

*Check with the distance:* $d(C,t) = \dfrac{|-3-12-10|}{\sqrt{9+16}} = \dfrac{25}{5} = 5 = r$. □

**Exercise 5.3.** Find the intersections of $\Gamma$ with the line $e: \ x - y + 12 = 0$.

*Solution. Step 1: solve the line for $y$.* $y = x + 12$.

*Step 2: substitute into $\Gamma$.*

$$\begin{aligned} & x^2 + (x+12)^2 + 2x - 6(x+12) - 15 = 0 \\ \Longrightarrow \quad & x^2 + x^2 + 24x + 144 + 2x - 6x - 72 - 15 = 0 \\ \Longrightarrow \quad & 2x^2 + 20x + 57 = 0. \end{aligned}$$

*Step 3: discriminant.* $\Delta = 20^2 - 4 \cdot 2 \cdot 57 = 400 - 456 = -56 < 0$.

*Conclusion.* The equation has no real solutions: **no intersection points**; the line is external to $\Gamma$.

*Check with the distance:* $d(C,e) = \dfrac{|-1-3+12|}{\sqrt{2}} = \dfrac{8}{\sqrt{2}} = 4\sqrt{2} \approx 5.66 > 5$. □

# 6 Intersections of a parabola and a circle

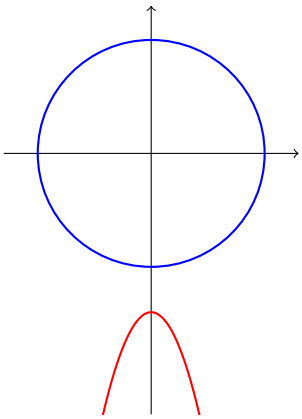
**Method.** In this sheet the parabolas open *downwards*: $y = q - x^2$. Replace $x^2$ with $q - y$ in the equation of the circle (center on the $y$-axis). We get a quadratic equation in $y$ only. Then, for each real root $\bar{y}$:

- if $\bar{y} < q$: $x^2 = q - \bar{y} > 0$ gives *two* points $(\pm\sqrt{q-\bar{y}},\, \bar{y})$;
- if $\bar{y} = q$: $x^2 = 0$ gives *one* point $(0, q)$ (the vertex);
- if $\bar{y} > q$: $x^2 = q - \bar{y} < 0$ gives *no* points.

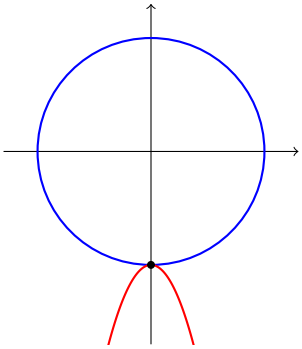
Note that the condition is now $\bar{y} \leq q$ (the parabola lies *below* its vertex).

**Non-symmetric configurations (Exercises 6.3–6.5).** When the axis of the parabola does not pass through the center of the circle, the trick above does not work. Substitute instead the whole expression $y = -x^2 + bx + c$ into the equation of the circle: we get a polynomial equation of degree 4 in $x$.

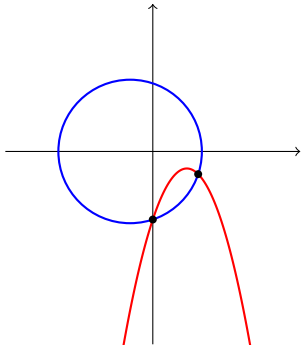
- Look for integer roots among the divisors of the constant term, and lower the degree by polynomial long division (or by factoring out $x$, or by grouping).
- Each *real* root $\bar{x}$ gives exactly *one* point $\big(\bar{x},\, -\bar{x}^2 + b\bar{x} + c\big)$.
- A *double* root corresponds to a point where the two curves are tangent.



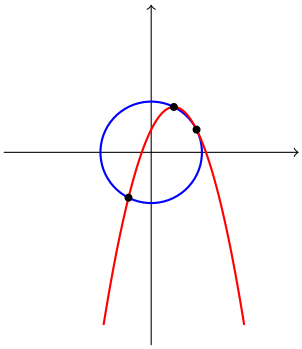
6.1: no points



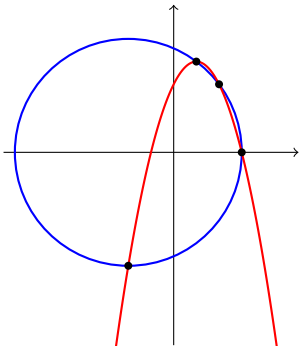
6.2: one point



6.3: two points



6.4: three points



6.5: four points

**Exercise 6.1.** Find the intersections of the circle $x^2 + y^2 = 25$ and the parabola $y = -x^2 - 7$.

*Solution. Step 1: express $x^2$ from the parabola.* $x^2 = -7 - y$, with the condition $y \leq -7$.

*Step 2: substitute into the circle.*

$$(-7 - y) + y^2 = 25 \implies y^2 - y - 32 = 0.$$

*Step 3: solve.* $\Delta = 1 + 128 = 129 > 0$, so there are two real roots

$$y_{1,2} = \frac{1 \pm \sqrt{129}}{2}, \qquad y_1 \approx 6.18, \quad y_2 \approx -5.18.$$

*Step 4: check the condition $y \leq -7$.* $y_1 > 0 > -7$ is clearly rejected. For $y_2$:

$$\frac{1 - \sqrt{129}}{2} \leq -7 \iff 1 - \sqrt{129} \leq -14 \iff \sqrt{129} \geq 15 \iff 129 \geq 225,$$

which is false, so $y_2 > -7$ is rejected too.

*Conclusion.* **No intersection points**, even though $\Delta > 0$: both roots give $x^2 = -7 - y < 0$.

*Quick check.* Every point of the parabola has $y = -x^2 - 7 \leq -7$, while every point of the circle has $|y| \leq 5$: the two curves cannot meet. □

**Exercise 6.2.** Find the intersections of the circle $x^2 + y^2 = 25$ and the parabola $y = -x^2 - 5$.

*Solution. Step 1.* $x^2 = -5 - y$, with the condition $y \leq -5$.

*Step 2: substitute.*

$$(-5 - y) + y^2 = 25 \implies y^2 - y - 30 = 0.$$

*Step 3: solve.* $\Delta = 1 + 120 = 121$, $\quad y = \dfrac{1 \pm 11}{2}$, so $y_1 = 6$, $y_2 = -5$.

*Step 4: check the condition $y \leq -5$ and find $x$.*

- $y_1 = 6 > -5$: $x^2 = -11$, impossible: no points;

- $y_2 = -5$: $x^2 = 0 \Rightarrow x = 0$: the point $(0, -5)$.

*Conclusion.* **One intersection point:** $(0, -5)$. The vertex of the parabola is the lowest point of the circle; the parabola opens downwards, away from the circle, and the two curves are tangent there. □

**Exercise 6.3.** Find the intersections of the circle $(x + 1)^2 + y^2 = 10$ and the parabola $y = -x^2 + 3x - 3$.

*Solution. Step 1: substitute y into the circle.* We need $(-x^2+3x-3)^2 = x^4-6x^3+15x^2-18x+9$ and $(x + 1)^2 = x^2 + 2x + 1$, so

$$\begin{aligned} & x^2 + 2x + 1 + x^4 - 6x^3 + 15x^2 - 18x + 9 = 10 \\ \Longrightarrow \quad & x^4 - 6x^3 + 16x^2 - 16x = 0. \end{aligned}$$

*Step 2: factor out x.* $x\,(x^3 - 6x^2 + 16x - 16) = 0$, so $x = 0$ is a root.

*Step 3: roots of the cubic.* Integer roots must divide 16: try $\pm1, \pm2, \ldots$ For $x = 1$: $1 - 6 + 16 - 16 = -5 \neq 0$; for $x = 2$: $8 - 24 + 32 - 16 = 0$. Divide by $x - 2$:

$$\begin{array}{rl|l} x^3 - 6x^2 + 16x - 16 & & x - 2 \\ \cline{3-3} -\,x^3 + 2x^2 & & x^2 - 4x + 8 \\ \cline{1-1} \quad -\,4x^2 + 16x & & \\ \quad\; 4x^2 \;\; - 8x & & \\ \cline{1-1} \qquad\qquad 8x - 16 & & \\ \qquad\qquad -\,8x + 16 & & \\ \cline{1-1} \qquad\qquad 0 & & \end{array}$$

so $x^3 - 6x^2 + 16x - 16 = (x - 2)(x^2 - 4x + 8)$.

*Step 4: the remaining quadratic.* $x^2 - 4x + 8 = 0$ has $\Delta = 16 - 32 = -16 < 0$: no real roots.

*Step 5: find y.* The real roots are $x = 0$ and $x = 2$:

$$x = 0: \; y = -3, \qquad x = 2: \; y = -4 + 6 - 3 = -1.$$

*Conclusion.* **Two intersection points:** $(0, -3)$ and $(2, -1)$.

*Check:* $(0 + 1)^2 + (-3)^2 = 10$ and $(2 + 1)^2 + (-1)^2 = 10$. □

**Exercise 6.4.** Find the intersections of the circle $x^2 + y^2 = 5$ and the parabola $y = -x^2 + 2x + 1$.

*Solution. Step 1: substitute y into the circle.* Since $(-x^2 + 2x + 1)^2 = x^4 - 4x^3 + 2x^2 + 4x + 1$,

$$\begin{aligned} & x^2 + x^4 - 4x^3 + 2x^2 + 4x + 1 = 5 \\ \Longrightarrow \quad & x^4 - 4x^3 + 3x^2 + 4x - 4 = 0. \end{aligned}$$

*Step 2: a first root.* Integer roots must divide 4: candidates $\pm1, \pm2, \pm4$. For $x = 1$: $1 - 4 + 3 + 4 - 4 = 0$. Divide by $x - 1$ (note the missing $x$ term in the quotient):

$$\begin{array}{rl|l} x^4 - 4x^3 + 3x^2 + 4x - 4 & & x - 1 \\ \cline{3-3} -\,x^4 \;\; + x^3 & & x^3 - 3x^2 + 4 \\ \cline{1-1} \quad -\,3x^3 + 3x^2 & & \\ \quad\; 3x^3 - 3x^2 & & \\ \cline{1-1} \qquad\qquad 4x - 4 & & \\ \qquad\qquad -\,4x + 4 & & \\ \cline{1-1} \qquad\qquad 0 & & \end{array}$$

so the equation becomes $(x - 1)(x^3 - 3x^2 + 4) = 0$.

*Step 3: a second root.* For $x=-1$: $-1-3+4=0$. Divide by $x+1$:

$$\begin{array}{rl|l} x^3-3x^2 & +4 & x+1 \\ \cline{3-3} -x^3\ -x^2 & & x^2-4x+4 \\ \cline{1-1} \quad -4x^2 & & \\ \quad 4x^2+4x & & \\ \cline{1-1} \qquad 4x+4 & & \\ \qquad -4x-4 & & \\ \cline{1-1} \qquad\quad 0 & & \end{array}$$

and $x^2-4x+4=(x-2)^2$. Hence

$$x^4-4x^3+3x^2+4x-4=(x-1)(x+1)(x-2)^2.$$

*Step 4: find $y$.*

$$x=1:\ y=2, \qquad x=-1:\ y=-1-2+1=-2, \qquad x=2:\ y=-4+4+1=1.$$

*Conclusion.* **Three intersection points:** $(1,2)$, $(-1,-2)$, $(2,1)$. The root $x=2$ is *double*: at $(2,1)$ the parabola and the circle are tangent. Unlike the symmetric case, the tangency point is not the vertex of the parabola (the vertex is $(1,2)$, where the curves simply cross).

*Check:* $1+4=5$, $1+4=5$, $4+1=5$. □

**Exercise 6.5.** Find the intersections of the circle $(x+2)^2+y^2=25$ and the parabola $y=-x^2+2x+3$.

*Solution. Step 1: substitute $y$ into the circle.* Since $(-x^2+2x+3)^2=x^4-4x^3-2x^2+12x+9$ and $(x+2)^2=x^2+4x+4$,

$$\begin{aligned} & x^2+4x+4+x^4-4x^3-2x^2+12x+9=25 \\ \Longrightarrow\quad & x^4-4x^3-x^2+16x-12=0. \end{aligned}$$

*Step 2: a first root.* Integer roots must divide 12. For $x=1$: $1-4-1+16-12=0$. Divide by $x-1$:

$$\begin{array}{rl|l} x^4-4x^3 & -x^2+16x-12 & x-1 \\ \cline{3-3} -x^4\ +x^3 & & x^3-3x^2-4x+12 \\ \cline{1-1} \quad -3x^3\ -x^2 & & \\ \quad 3x^3-3x^2 & & \\ \cline{1-1} \qquad -4x^2+16x & & \\ \qquad 4x^2\ -4x & & \\ \cline{1-1} \qquad\quad 12x-12 & & \\ \qquad\quad -12x+12 & & \\ \cline{1-1} \qquad\qquad 0 & & \end{array}$$

so the equation becomes $(x-1)(x^3-3x^2-4x+12)=0$.

*Step 3: factor the cubic by grouping.*

$$x^3-3x^2-4x+12=x^2(x-3)-4(x-3)=(x-3)(x^2-4)=(x-3)(x-2)(x+2).$$

Hence the four roots are $x=1,\ 2,\ 3,\ -2$.

*Step 4: find $y$.*

$$x=1:\ y=4, \qquad x=2:\ y=3, \qquad x=3:\ y=0, \qquad x=-2:\ y=-4-4+3=-5.$$

*Conclusion.* **Four intersection points:** $(1,4)$, $(2,3)$, $(3,0)$, $(-2,-5)$. Note that, without symmetry, the points no longer come in pairs $(\pm\bar{x},\bar{y})$.

*Check:* $9+16$, $16+9$, $25+0$, $0+25$ all equal 25. □