

Analytic Geometry: Points, Lines, Parabolas and Circles

Exercises with step-by-step solutions

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1 Distance between two points

Reminder. The distance between $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Exercise 1.1. Compute the distance between $A(1, 2)$ and $B(4, 6)$.

Solution. Step 1: differences of coordinates. $x_2 - x_1 = 4 - 1 = 3$, $y_2 - y_1 = 6 - 2 = 4$.

Step 2: squares and sum. $3^2 + 4^2 = 9 + 16 = 25$.

Step 3: square root. $d(A, B) = \sqrt{25} = 5$. □

Exercise 1.2. Compute the distance between $P(2, -1)$ and $Q(-3, 4)$.

Solution. Step 1. $x_Q - x_P = -3 - 2 = -5$, $y_Q - y_P = 4 - (-1) = 5$.

Step 2. $(-5)^2 + 5^2 = 25 + 25 = 50$.

Step 3. $d(P, Q) = \sqrt{50} = \sqrt{25 \cdot 2} = 5\sqrt{2} \approx 7.07$.

Note that the sign of the differences does not matter, since they are squared. □

2 Circle through three points

Method. Write the circle in the general form

$$x^2 + y^2 + ax + by + c = 0.$$

Substituting the coordinates of the three points gives a linear system in a, b, c . If the system has a (unique) solution, the circle has

$$\text{center } C\left(-\frac{a}{2}, -\frac{b}{2}\right), \quad r = \sqrt{\frac{a^2}{4} + \frac{b^2}{4} - c}.$$

If the three points are collinear the system is impossible: no circle passes through them.

Exercise 2.1. Find the circle through $A(5, 7)$, $B(-3, 3)$, $D(2, -2)$.

Solution. Step 1: substitute the points.

$$\begin{array}{lll} A(5, 7) : & 25 + 49 + 5a + 7b + c = 0 & \implies 5a + 7b + c = -74 \\ B(-3, 3) : & 9 + 9 - 3a + 3b + c = 0 & \implies -3a + 3b + c = -18 \\ D(2, -2) : & 4 + 4 + 2a - 2b + c = 0 & \implies 2a - 2b + c = -8 \end{array}$$

Step 2: eliminate c by subtracting the third equation from the other two:

$$\begin{array}{lll} (5a + 7b + c) - (2a - 2b + c) = -74 + 8 & \implies & 3a + 9b = -66 \implies a + 3b = -22, \\ (-3a + 3b + c) - (2a - 2b + c) = -18 + 8 & \implies & -5a + 5b = -10 \implies b = a - 2. \end{array}$$

Step 3: solve. $a + 3(a - 2) = -22 \Rightarrow 4a = -16 \Rightarrow a = -4$, hence $b = -6$. From the third equation: $c = -8 - 2a + 2b = -8 + 8 - 12 = -12$.

Step 4: equation, center, radius.

$$x^2 + y^2 - 4x - 6y - 12 = 0, \quad C(2, 3), \quad r = \sqrt{4 + 9 + 12} = \sqrt{25} = 5.$$

Check: $d(C, A) = \sqrt{3^2 + 4^2} = 5$, $d(C, B) = \sqrt{5^2 + 0^2} = 5$, $d(C, D) = \sqrt{0^2 + 5^2} = 5$. □

Exercise 2.2. Find the circle through $A(0, 5)$, $B(2, 3)$, $D(-4, 1)$.

Solution. Step 1: substitute the points.

$$\begin{array}{lll} A(0, 5) : & 0 + 25 + 0 + 5b + c = 0 & \implies 5b + c = -25 \\ B(2, 3) : & 4 + 9 + 2a + 3b + c = 0 & \implies 2a + 3b + c = -13 \\ D(-4, 1) : & 16 + 1 - 4a + b + c = 0 & \implies -4a + b + c = -17 \end{array}$$

Step 2: eliminate c by subtracting the first equation from the other two:

$$\begin{array}{ll} 2a - 2b = 12 & \implies a - b = 6, \\ -4a - 4b = 8 & \implies a + b = -2. \end{array}$$

Step 3: solve. Adding: $2a = 4 \Rightarrow a = 2$, so $b = -4$. Then $c = -25 - 5b = -25 + 20 = -5$.

Step 4: equation, center, radius.

$$x^2 + y^2 + 2x - 4y - 5 = 0, \quad C(-1, 2), \quad r = \sqrt{1 + 4 + 5} = \sqrt{10}.$$

Check: $d(C, A)^2 = 1^2 + 3^2 = 10$, $d(C, B)^2 = 3^2 + 1^2 = 10$, $d(C, D)^2 = (-3)^2 + (-1)^2 = 10$. □

Exercise 2.3. Find, if it exists, the circle through $A(1, 1)$, $B(3, 5)$, $D(-2, -5)$.

Solution. Step 1: substitute the points.

$$\begin{array}{lll} A(1, 1) : & 2 + a + b + c = 0 & \implies a + b + c = -2 \\ B(3, 5) : & 34 + 3a + 5b + c = 0 & \implies 3a + 5b + c = -34 \\ D(-2, -5) : & 29 - 2a - 5b + c = 0 & \implies -2a - 5b + c = -29 \end{array}$$

Step 2: eliminate c by subtracting the first equation from the other two:

$$\begin{array}{ll} 2a + 4b = -32 & \implies a + 2b = -16, \\ -3a - 6b = -27 & \implies a + 2b = 9. \end{array}$$

Step 3: conclusion. The quantity $a + 2b$ cannot be both -16 and 9 : the system is **impossible**, so **no circle** passes through the three points.

Geometric reason. The slopes are $m_{AB} = \frac{5-1}{3-1} = 2$ and $m_{AD} = \frac{-5-1}{-2-1} = 2$: the points are collinear (they lie on the line $y = 2x - 1$), and three distinct collinear points never lie on a circle. □

3 Circle with given center and radius

Reminder. The circle with center $C(x_0, y_0)$ and radius $r > 0$ is

$$(x - x_0)^2 + (y - y_0)^2 = r^2.$$

Exercise 3.1. Write the equation of the circle with center $C(3, -2)$ and radius $r = 4$.

Solution. Step 1: standard form. $(x - 3)^2 + (y + 2)^2 = 16$.

Step 2: expand. $x^2 - 6x + 9 + y^2 + 4y + 4 = 16$.

Step 3: general form. $x^2 + y^2 - 6x + 4y - 3 = 0$.

Check: $a = -6$, $b = 4$, $c = -3$ give $C(3, -2)$ and $r = \sqrt{9 + 4 + 3} = 4$. □

Exercise 3.2. Write the equation of the circle with center $C(-\frac{1}{2}, 3)$ and radius $r = \sqrt{5}$.

Solution. Step 1: standard form. $(x + \frac{1}{2})^2 + (y - 3)^2 = 5$.

Step 2: expand. $x^2 + x + \frac{1}{4} + y^2 - 6y + 9 = 5$.

Step 3: general form. $x^2 + y^2 + x - 6y + \frac{17}{4} = 0$, or, multiplying by 4,

$$4x^2 + 4y^2 + 4x - 24y + 17 = 0.$$

Check: $a = 1$, $b = -6$, $c = \frac{17}{4}$ give $C(-\frac{1}{2}, 3)$ and $r^2 = \frac{1}{4} + 9 - \frac{17}{4} = 5$. □

4 Center and radius from the equation

Method. An equation $\alpha x^2 + \alpha y^2 + \dots = 0$ (same coefficient of x^2 and y^2 , no xy term) is first divided by α to get $x^2 + y^2 + ax + by + c = 0$. Completing the squares,

$$(x + \frac{a}{2})^2 + (y + \frac{b}{2})^2 = \frac{a^2}{4} + \frac{b^2}{4} - c.$$

It is a circle only if the right-hand side is *positive*.

Exercise 4.1. Find center and radius of $x^2 + y^2 - 8x + 2y + 8 = 0$.

Solution. Step 1: group. $(x^2 - 8x) + (y^2 + 2y) + 8 = 0$.

Step 2: complete the squares. $x^2 - 8x = (x - 4)^2 - 16$, $y^2 + 2y = (y + 1)^2 - 1$, so

$$(x - 4)^2 - 16 + (y + 1)^2 - 1 + 8 = 0 \implies (x - 4)^2 + (y + 1)^2 = 9.$$

Step 3: read off. Since $9 > 0$: center $C(4, -1)$, radius $r = 3$.

With the formulas: $a = -8$, $b = 2$, $c = 8$: $C(4, -1)$, $r^2 = 16 + 1 - 8 = 9$. □

Exercise 4.2. Find center and radius of $2x^2 + 2y^2 + 6x - 10y - 1 = 0$.

Solution. Step 1: normalize. The coefficients of x^2 and y^2 are equal; divide by 2:

$$x^2 + y^2 + 3x - 5y - \frac{1}{2} = 0.$$

Step 2: complete the squares. $x^2 + 3x = (x + \frac{3}{2})^2 - \frac{9}{4}$, $y^2 - 5y = (y - \frac{5}{2})^2 - \frac{25}{4}$, so

$$(x + \frac{3}{2})^2 + (y - \frac{5}{2})^2 = \frac{9}{4} + \frac{25}{4} + \frac{1}{2} = \frac{36}{4} = 9.$$

Step 3: read off. Center $C(-\frac{3}{2}, \frac{5}{2})$, radius $r = 3$.

Warning: applying the formulas to the *non-divided* equation would give a wrong result. □

Exercise 4.3. Find, if possible, center and radius of $x^2 + y^2 - 2x + 4y + 10 = 0$.

Solution. Step 1: complete the squares. $x^2 - 2x = (x - 1)^2 - 1$, $y^2 + 4y = (y + 2)^2 - 4$, so

$$(x - 1)^2 - 1 + (y + 2)^2 - 4 + 10 = 0 \implies (x - 1)^2 + (y + 2)^2 = -5.$$

Step 2: conclusion. The left-hand side is a sum of squares, hence ≥ 0 for every real (x, y) , while the right-hand side is $-5 < 0$. No real point satisfies the equation: it is **not a circle** (the “radius” would be $\sqrt{-5}$, not a real number).

With the formula: $r^2 = \frac{a^2}{4} + \frac{b^2}{4} - c = 1 + 4 - 10 = -5 < 0$. □

5 Intersections of a line and a circle

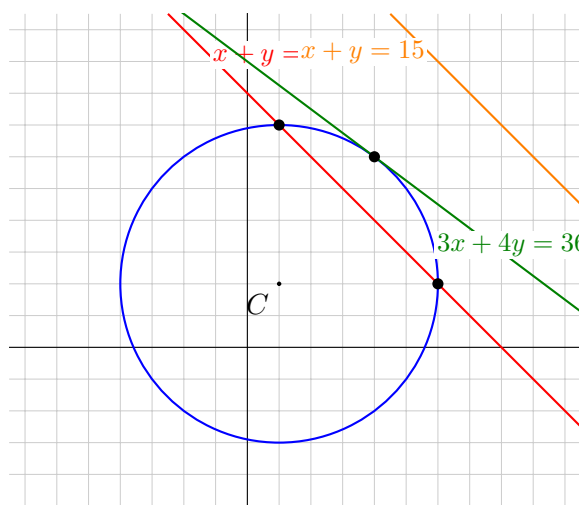
Method. Solve the system circle–line by substitution: one obtains a quadratic equation, whose discriminant Δ tells us the number of intersection points:

$\Delta > 0$: 2 points (secant), $\Delta = 0$: 1 point (tangent), $\Delta < 0$: no points (external line).

Equivalently, compare the distance d of the center from the line with the radius r : $d < r$, $d = r$, $d > r$ respectively. Recall $d(P_0, \alpha x + \beta y + \gamma = 0) = \frac{|\alpha x_0 + \beta y_0 + \gamma|}{\sqrt{\alpha^2 + \beta^2}}$.

In all three exercises the circle is

$$\Gamma : x^2 + y^2 - 2x - 4y - 20 = 0, \quad \text{center } C(1, 2), \quad r = \sqrt{1 + 4 + 20} = 5.$$



Exercise 5.1. Find the intersections of Γ with the line $s : x + y - 8 = 0$.

Solution. Step 1: solve the line for y . $y = 8 - x$.

Step 2: substitute into Γ .

$$\begin{aligned} x^2 + (8 - x)^2 - 2x - 4(8 - x) - 20 &= 0 \\ \implies x^2 + 64 - 16x + x^2 - 2x - 32 + 4x - 20 &= 0 \\ \implies 2x^2 - 14x + 12 &= 0. \end{aligned}$$

Step 3: solve. Dividing by 2: $x^2 - 7x + 6 = 0$, with $\Delta = 49 - 24 = 25 > 0$:

$$x = \frac{7 \pm 5}{2} \implies x_1 = 1, \quad x_2 = 6.$$

Step 4: find y . $y_1 = 8 - 1 = 7$, $y_2 = 8 - 6 = 2$. **Two intersection points:** $(1, 7)$ and $(6, 2)$; the line is secant.

Check with the distance: $d(C, s) = \frac{|1 + 2 - 8|}{\sqrt{2}} = \frac{5}{\sqrt{2}} \approx 3.54 < 5$. □

Exercise 5.2. Find the intersections of Γ with the line $t : 3x + 4y - 36 = 0$.

Solution. Step 1: solve the line for y . $y = \frac{36 - 3x}{4}$.

Step 2: substitute into Γ and multiply everything by 16 to clear denominators:

$$\begin{aligned} 16x^2 + (36 - 3x)^2 - 32x - 16(36 - 3x) - 320 &= 0, \\ 16x^2 + 1296 - 216x + 9x^2 - 32x - 576 + 48x - 320 &= 0 \implies 25x^2 - 200x + 400 = 0. \end{aligned}$$

Step 3: solve. Dividing by 25: $x^2 - 8x + 16 = 0$, i.e. $(x - 4)^2 = 0$, so $\Delta = 0$ and $x = 4$ (double root).

Step 4: find y. $y = \frac{36 - 12}{4} = 6$. **One intersection point:** $(4, 6)$; the line is tangent to Γ .

Check with the distance: $d(C, t) = \frac{|3 + 8 - 36|}{\sqrt{9 + 16}} = \frac{25}{5} = 5 = r$. $\square$

Exercise 5.3. Find the intersections of Γ with the line $e : x + y - 15 = 0$.

Solution. Step 1: solve the line for y. $y = 15 - x$.

Step 2: substitute into Γ .

$$\begin{aligned} x^2 + (15 - x)^2 - 2x - 4(15 - x) - 20 &= 0 \\ \implies x^2 + 225 - 30x + x^2 - 2x - 60 + 4x - 20 &= 0 \\ \implies 2x^2 - 28x + 145 &= 0. \end{aligned}$$

Step 3: discriminant. $\Delta = 28^2 - 4 \cdot 2 \cdot 145 = 784 - 1160 = -376 < 0$.

Conclusion. The equation has no real solutions: **no intersection points**; the line is external to Γ .

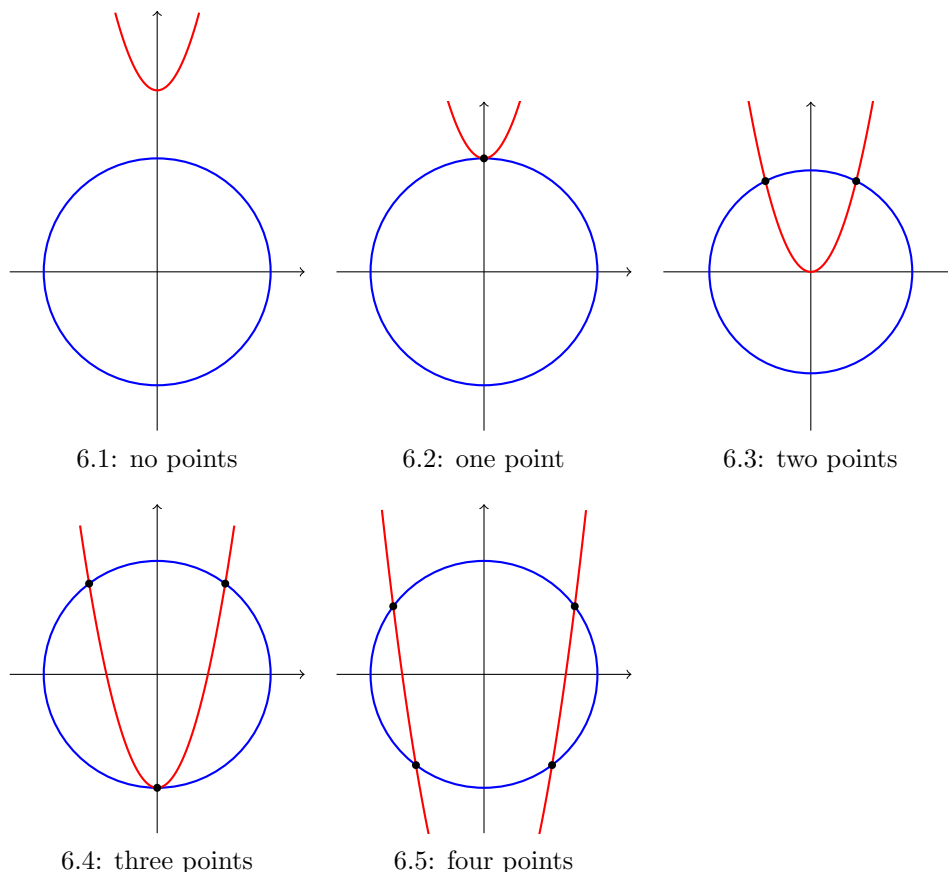
Check with the distance: $d(C, e) = \frac{|1 + 2 - 15|}{\sqrt{2}} = \frac{12}{\sqrt{2}} = 6\sqrt{2} \approx 8.49 > 5$. $\square$

6 Intersections of a parabola and a circle

Method. For a parabola $y = x^2 + q$ and a circle $x^2 + y^2 + by + c = 0$ (center on the y -axis), replace x^2 with $y - q$ in the equation of the circle. We get a quadratic equation in y only. Then, for each real root $\bar{y}$:

- if $\bar{y} > q$: $x^2 = \bar{y} - q > 0$ gives *two* points $(\pm\sqrt{\bar{y} - q}, \bar{y})$;
- if $\bar{y} = q$: $x^2 = 0$ gives *one* point $(0, q)$ (the vertex);
- if $\bar{y} < q$: $x^2 = \bar{y} - q < 0$ gives *no* points.

Forgetting the condition $\bar{y} \geq q$ is the most common mistake!



Exercise 6.1. Find the intersections of the circle $x^2 + y^2 = 25$ and the parabola $y = x^2 + 8$.

Solution. Step 1: express x^2 from the parabola. $x^2 = y - 8$, with the condition $y \geq 8$.

Step 2: substitute into the circle.

$$(y - 8) + y^2 = 25 \implies y^2 + y - 33 = 0.$$

Step 3: solve. $\Delta = 1 + 132 = 133 > 0$, so there are two real roots

$$y_{1,2} = \frac{-1 \pm \sqrt{133}}{2}, \quad y_1 \approx 5.27, \quad y_2 \approx -6.27.$$

Step 4: check the condition $y \geq 8$. $y_2 < 0 < 8$ is clearly rejected. For y_1 :

$$\frac{-1 + \sqrt{133}}{2} \geq 8 \iff \sqrt{133} \geq 17 \iff 133 \geq 289,$$

which is false, so $y_1 < 8$ is rejected too.

Conclusion. No intersection points, even though $\Delta > 0$: both roots give $x^2 = y - 8 < 0$.

Quick check. Every point of the parabola has $y = x^2 + 8 \geq 8$, while every point of the circle has $|y| \leq 5$: the two curves cannot meet. $\square$

Exercise 6.2. Find the intersections of the circle $x^2 + y^2 = 25$ and the parabola $y = x^2 + 5$.

Solution. Step 1. $x^2 = y - 5$, with the condition $y \geq 5$.

Step 2: substitute.

$$(y - 5) + y^2 = 25 \implies y^2 + y - 30 = 0.$$

Step 3: solve. $\Delta = 1 + 120 = 121$, $y = \frac{-1 \pm 11}{2}$, so $y_1 = 5$, $y_2 = -6$.

Step 4: check the condition $y \geq 5$ and find x .

- $y_1 = 5$: $x^2 = 0 \implies x = 0$: the point $(0, 5)$;

- $y_2 = -6 < 5$: $x^2 = -11$, impossible: no points.

Conclusion. **One intersection point:** $(0, 5)$. The vertex of the parabola is the top point of the circle, and the two curves are tangent there. $\square$

Exercise 6.3. Find the intersections of the circle $x^2 + y^2 = 20$ and the parabola $y = x^2$.

Solution. *Step 1.* $x^2 = y$, with the condition $y \geq 0$.

Step 2: substitute.

$$y + y^2 = 20 \implies y^2 + y - 20 = 0.$$

Step 3: solve. $\Delta = 1 + 80 = 81$, $y = \frac{-1 \pm 9}{2}$, so $y_1 = 4$, $y_2 = -5$.

Step 4: check the condition $y \geq 0$ and find x .

- $y_1 = 4 > 0$: $x^2 = 4 \Rightarrow x = \pm 2$: the points $(2, 4)$, $(-2, 4)$;
- $y_2 = -5 < 0$: $x^2 = -5$, impossible: no points.

Conclusion. **Two intersection points:** $(2, 4)$ and $(-2, 4)$. $\square$

Exercise 6.4. Find the intersections of the circle $x^2 + y^2 = 25$ and the parabola $y = x^2 - 5$.

Solution. *Step 1.* $x^2 = y + 5$, with the condition $y \geq -5$.

Step 2: substitute.

$$(y + 5) + y^2 = 25 \implies y^2 + y - 20 = 0.$$

Step 3: solve. $\Delta = 81$, $y = \frac{-1 \pm 9}{2}$, so $y_1 = 4$, $y_2 = -5$.

Step 4: check the condition $y \geq -5$ and find x .

- $y_1 = 4 > -5$: $x^2 = 9 \Rightarrow x = \pm 3$: the points $(3, 4)$, $(-3, 4)$;
- $y_2 = -5$: $x^2 = 0 \Rightarrow x = 0$: the point $(0, -5)$.

Conclusion. **Three intersection points:** $(3, 4)$, $(-3, 4)$, $(0, -5)$. The vertex of the parabola is the lowest point of the circle (a tangency point), and the parabola crosses the circle again at two symmetric points. $\square$

Exercise 6.5. Find the intersections of the circle $x^2 + y^2 = 25$ and the parabola $y = x^2 - 13$.

Solution. *Step 1.* $x^2 = y + 13$, with the condition $y \geq -13$.

Step 2: substitute.

$$(y + 13) + y^2 = 25 \implies y^2 + y - 12 = 0.$$

Step 3: solve. $\Delta = 1 + 48 = 49$, $y = \frac{-1 \pm 7}{2}$, so $y_1 = 3$, $y_2 = -4$.

Step 4: check the condition $y \geq -13$ and find x .

- $y_1 = 3 > -13$: $x^2 = 16 \Rightarrow x = \pm 4$: the points $(4, 3)$, $(-4, 3)$;
- $y_2 = -4 > -13$: $x^2 = 9 \Rightarrow x = \pm 3$: the points $(3, -4)$, $(-3, -4)$.

Conclusion. **Four intersection points:** $(4, 3)$, $(-4, 3)$, $(3, -4)$, $(-3, -4)$. $\square$