Lines in the Cartesian Plane: Rules and Step-by-Step Examples

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Convention

Every line in the plane is written in exactly one of the two forms

$$\textbf{(non-vertical)} \quad y = mx + n \qquad\qquad \textbf{(vertical)} \quad x = a.$$

- In $y = mx + n$ the number m is the *slope* and n is the *y-intercept*: the line meets the y-axis at the point $(0, n)$.
- If $m = 0$ the line $y = n$ is *horizontal*.
- A vertical line $x = a$ has *no slope*: it cannot be written as $y = mx + n$. It meets the x-axis at the point $(a, 0)$.

The first thing to do in every exercise is therefore to decide whether the line we are looking for is vertical or not.

1 A point belongs to a line

Rules

- (x_0, y_0) belongs to $y = mx + n$ if and only if $y_0 = mx_0 + n$ (substitute the coordinates and check that the equality holds).
- (x_0, y_0) belongs to $x = a$ if and only if $x_0 = a$ (the value of y_0 does not matter).
- *Line with slope m through (x_0, y_0):* $y = m(x - x_0) + y_0$.
- *Line through two distinct points (x_1, y_1) and (x_2, y_2):*
  - if $x_1 = x_2$ the line is vertical: $x = x_1$;
  - if $x_1 \neq x_2$ the slope is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ and the line is $y = m(x - x_1) + y_1$.

Examples

Example 1.1. Decide whether the points $A = (2, 7)$ and $B = (1, 5)$ belong to the line $r : \ y = 3x+1$.

Step 1. The line is non-vertical, so a point (x_0, y_0) belongs to it if and only if $y_0 = 3x_0 + 1$.

Step 2. Point $A = (2, 7)$: $x_0 = 2$, $y_0 = 7$. Compute the right-hand side: $3 \cdot 2 + 1 = 7$. The left-hand side is $y_0 = 7$. Since $7 = 7$, $A \in r$.

Step 3. Point $B = (1, 5)$: $x_0 = 1$, $y_0 = 5$. Compute the right-hand side: $3 \cdot 1 + 1 = 4$. The left-hand side is $y_0 = 5$. Since $5 \neq 4$, $B \notin r$.

Answer: $A \in r, \;\; B \notin r$.

Example 1.2. Find the line through $P = (1, 3)$ and $Q = (4, 9)$, and the line through $R = (2, -1)$ and $S = (2, 6)$.

Step 1. Line PQ: compare the x-coordinates: $1 \neq 4$, so the line is non-vertical and has the form $y = mx + n$.

Step 2. Compute the slope:

$$m = \frac{y_Q - y_P}{x_Q - x_P} = \frac{9 - 3}{4 - 1} = \frac{6}{3} = 2.$$

Step 3. Impose the passage through $P = (1, 3)$: $y = 2(x - 1) + 3 = 2x - 2 + 3$, that is $y = 2x + 1$.

Step 4. Check with $Q = (4, 9)$: $2 \cdot 4 + 1 = 9$. ✓

Step 5. Line RS: compare the x-coordinates: $2 = 2$, so the line is vertical: $x = 2$.

Step 6. Check: both R and S have x-coordinate 2. ✓

Answer: $PQ: \;\; y = 2x + 1, \qquad RS: \;\; x = 2.$

2 Intersection of two lines

Rules

The common points of two lines are the solutions of the system formed by their two equations.

- $y = m_1x+n_1$ and $y = m_2x+n_2$ (both non-vertical): set the two right-hand sides equal, $m_1x+n_1 = m_2x + n_2$.
  - if $m_1 \neq m_2$: exactly one common point, with $x = \dfrac{n_2 - n_1}{m_1 - m_2}$ and $y = m_1x + n_1$;
  - if $m_1 = m_2$ and $n_1 \neq n_2$: no common point (the lines are parallel);
  - if $m_1 = m_2$ and $n_1 = n_2$: the lines coincide.
- $x = a$ and $y = mx + n$: always exactly one common point, obtained by substituting $x = a$: the point $(a, \; ma + n)$.
- $x = a$ and $x = b$: no common point if $a \neq b$; the same line if $a = b$.

Examples

Example 2.1. Find the intersection of $r: \;\; y = 2x + 1$ and $s: \;\; y = -x + 7$.

Step 1. Both lines are non-vertical, with slopes $m_1 = 2$ and $m_2 = -1$. Since $2 \neq -1$ there is exactly one intersection point.

Step 2. Write the system and set the two expressions for y equal:

$$\begin{cases} y = 2x + 1 \\ y = -x + 7 \end{cases} \qquad \Longrightarrow \qquad 2x + 1 = -x + 7.$$

Step 3. Solve for x: $2x + x = 7 - 1$, so $3x = 6$ and $x = 2$.

Step 4. Substitute $x = 2$ into one of the equations, e.g. the first: $y = 2 \cdot 2 + 1 = 5$.

Step 5. Check in the other equation: $-2+7=5$. ✓

Answer: $r \cap s = \{(2,5)\}$.

Example 2.2. Find the intersection of $r:\ x=3$ and $s:\ y=4x-5$, and of r with $t:\ x=-1$.

Step 1. r is vertical and s is non-vertical: there is exactly one common point.

Step 2. Every point of r has $x=3$. Substitute $x=3$ in the equation of s: $y=4\cdot 3-5=12-5=7$.

Step 3. So $r \cap s = \{(3,7)\}$. Check: $x=3$ satisfies r, and $4\cdot 3-5=7$ satisfies s. ✓

Step 4. $r:\ x=3$ and $t:\ x=-1$ are both vertical with $3 \neq -1$: a point cannot have $x=3$ and $x=-1$ at the same time, so there is no common point.

Answer: $r \cap s = \{(3,7)\}$, $\qquad r \cap t = \emptyset$.

3 Parallel lines

Rules

- $y=m_1x+n_1$ and $y=m_2x+n_2$ are parallel if and only if $m_1=m_2$ (same slope).
- Any two vertical lines $x=a$ and $x=b$ are parallel.
- A vertical line is never parallel to a non-vertical line.
- *Line parallel to $y=mx+n$ through (x_0,y_0):* $y=m(x-x_0)+y_0$.
- *Line parallel to $x=a$ through (x_0,y_0):* $x=x_0$.

Examples

Example 3.1. Find the line r parallel to $s:\ y=3x-2$ and passing through $P=(1,4)$.

Step 1. s is non-vertical, so r is non-vertical too: $r:\ y=mx+n$.

Step 2. Parallel lines have the same slope. The slope of s is 3, hence $m=3$ and $r:\ y=3x+n$.

Step 3. Impose $P=(1,4)\in r$: $4=3\cdot 1+n$, so $n=4-3=1$.

Step 4. Hence $r:\ y=3x+1$.

Step 5. Check: $3\cdot 1+1=4$, so $P\in r$; and $1\neq -2$, so r and s are distinct parallel lines. ✓

Answer: $r:\ y=3x+1$.

Example 3.2. Find the line r parallel to $s:\ x=5$ and passing through $P=(-2,3)$.

Step 1. s is vertical, so every line parallel to it is vertical: $r:\ x=a$.

Step 2. Impose $P=(-2,3)\in r$: the x-coordinate of P must equal a, so $a=-2$.

Step 3. Hence $r:\ x=-2$.

Step 4. Check: P has x-coordinate -2. ✓

Answer: $r:\ x=-2$.

4 Perpendicular lines

Rules

- If $m_1 \neq 0$ and $m_2 \neq 0$: $y = m_1x + n_1$ and $y = m_2x + n_2$ are perpendicular if and only if $m_1m_2 = -1$, that is $m_2 = -\dfrac{1}{m_1}$.
- A horizontal line $y = n$ and a vertical line $x = a$ are always perpendicular.
- *Line perpendicular to $y = mx + n$, $m \neq 0$, through (x_0, y_0):* $y = -\dfrac{1}{m}(x - x_0) + y_0$.
- *Line perpendicular to the horizontal line $y = n$ through (x_0, y_0):* $x = x_0$.
- *Line perpendicular to the vertical line $x = a$ through (x_0, y_0):* $y = y_0$.

Examples

Example 4.1. Find the line r perpendicular to $s:\ y = 2x + 3$ and passing through $P = (4, 1)$.

Step 1. s is non-vertical and non-horizontal (slope $2 \neq 0$), so r is non-vertical as well: $r:\ y = mx+n$.

Step 2. Perpendicularity condition: $2 \cdot m = -1$, hence $m = -\dfrac{1}{2}$, and $r:\ y = -\dfrac{1}{2}x + n$.

Step 3. Impose $P = (4, 1) \in r$: $1 = -\dfrac{1}{2} \cdot 4 + n = -2 + n$, so $n = 3$.

Step 4. Hence $r:\ y = -\dfrac{1}{2}x + 3$.

Step 5. Check: $-\dfrac{1}{2} \cdot 4 + 3 = 1$, so $P \in r$; and $2 \cdot \left(-\dfrac{1}{2}\right) = -1$. ✓

Answer: $r:\ y = -\frac{1}{2}x + 3$.

Example 4.2. Find the line r perpendicular to $s:\ x = 1$ through $P = (2, 5)$, and the line t perpendicular to $u:\ y = -4$ through the same point P.

Step 1. $s:\ x = 1$ is vertical; the lines perpendicular to a vertical line are the horizontal ones, so $r:\ y = n$ (slope 0).

Step 2. Impose $P = (2, 5) \in r$: the y-coordinate of P must equal n, so $n = 5$ and $r:\ y = 5$.

Step 3. $u:\ y = -4$ is horizontal; the lines perpendicular to a horizontal line are the vertical ones, so $t:\ x = a$.

Step 4. Impose $P = (2, 5) \in t$: the x-coordinate of P must equal a, so $a = 2$ and $t:\ x = 2$.

Step 5. Check: P has y-coordinate 5 and x-coordinate 2. ✓

Answer: $r:\ y = 5$, $\quad t:\ x = 2$.

Summary

Condition	$y = m_1x + n_1$ and $y = m_2x + n_2$	involving vertical lines
$(x_0, y_0) \in r$	$y_0 = mx_0 + n$	$x_0 = a$
One intersection point	$m_1 \neq m_2$	$x = a$ meets $y = mx + n$ at $(a, ma + n)$
Parallel	$m_1 = m_2$	$x = a \parallel x = b$
Perpendicular	$m_1m_2 = -1$	$x = a \perp y = n$