

Polynomial Equations and Straight Lines

Exercises with Step-by-Step Solutions

Caboara

1 Polynomial equations

Exercise 1.

Solve the system

$$\begin{cases} 6x^3 - 13x^2 - 3x + 6 = 0 \\ 3x^3 - 8x^2 - 2x + 4 = 0 \end{cases}$$

Solution. We are looking for the values of x that are roots of *both* polynomials

$$P(x) = 6x^3 - 13x^2 - 3x + 6, \quad Q(x) = 3x^3 - 8x^2 - 2x + 4.$$

Step 1: eliminate the cubic term. If x solves both equations, then it also solves any combination of them. Since the leading coefficient of P is twice that of Q , we compute $P - 2Q$:

$$\begin{aligned} P(x) - 2Q(x) &= (6x^3 - 13x^2 - 3x + 6) - (6x^3 - 16x^2 - 4x + 8) \\ &= 3x^2 + x - 2. \end{aligned}$$

So every solution of the system must satisfy $3x^2 + x - 2 = 0$.

Step 2: solve the quadratic.

$$x = \frac{-1 \pm \sqrt{1 + 24}}{6} = \frac{-1 \pm 5}{6} \implies x = \frac{2}{3} \text{ or } x = -1.$$

Indeed $3x^2 + x - 2 = (3x - 2)(x + 1)$.

Step 3: check the candidates in one of the original equations. The candidates are only *necessary* conditions: we must check them.

- $x = -1$: $Q(-1) = -3 - 8 + 2 + 4 = -5 \neq 0$. Rejected.
- $x = \frac{2}{3}$: $Q(\frac{2}{3}) = 3 \cdot \frac{8}{27} - 8 \cdot \frac{4}{9} - 2 \cdot \frac{2}{3} + 4 = \frac{8}{9} - \frac{32}{9} - \frac{12}{9} + \frac{36}{9} = 0$. Accepted.

Since $P = (P - 2Q) + 2Q$ and both $P - 2Q$ and Q vanish at $x = \frac{2}{3}$, also $P(\frac{2}{3}) = 0$. (Direct check: $\frac{16}{9} - \frac{52}{9} - \frac{18}{9} + \frac{54}{9} = 0$.)

Step 4 (verification by factorisation). Since $x = \frac{2}{3}$ is a root of both, $3x - 2$ divides both polynomials.

$$\begin{array}{r|l} 6x^3 - 13x^2 - 3x + 6 & 3x - 2 \\ -6x^3 + 4x^2 & 2x^2 - 3x - 3 \\ \hline & -9x^2 - 3x \\ & 9x^2 - 6x \\ \hline & -9x + 6 \\ & 9x - 6 \\ \hline & 0 \end{array}$$

so $P(x) = (3x - 2)(2x^2 - 3x - 3)$, and

$$\begin{array}{r|l}
 3x^3 - 8x^2 - 2x + 4 & 3x - 2 \\
 - 3x^3 + 2x^2 & \hline
 - 6x^2 - 2x & \\
 6x^2 - 4x & \\
 \hline
 - 6x + 4 & \\
 6x - 4 & \\
 \hline
 0 &
 \end{array}$$

so $Q(x) = (3x - 2)(x^2 - 2x - 2)$.

The remaining roots are

$$2x^2 - 3x - 3 = 0 \iff x = \frac{3 \pm \sqrt{33}}{4}, \quad x^2 - 2x - 2 = 0 \iff x = 1 \pm \sqrt{3},$$

and these sets have no element in common. Hence the system has exactly one solution:

$$\boxed{x = \frac{2}{3}}.$$

□

Exercise 2.

Solve the equation $2x^3 + x^2 + x - 1 = 0$.

Solution. **Step 1: look for rational roots.** If $x = \frac{p}{q}$ (in lowest terms) is a rational root of a polynomial with integer coefficients, then p divides the constant term -1 and q divides the leading coefficient 2 . The candidates are therefore

$$x \in \{\pm 1, \pm \frac{1}{2}\}.$$

Step 2: test the candidates. Let $P(x) = 2x^3 + x^2 + x - 1$.

$$P(1) = 2 + 1 + 1 - 1 = 3 \neq 0,$$

$$P(-1) = -2 + 1 - 1 - 1 = -3 \neq 0,$$

$$P(\frac{1}{2}) = \frac{2}{8} + \frac{1}{4} + \frac{1}{2} - 1 = \frac{1}{4} + \frac{1}{4} + \frac{1}{2} - 1 = 0.$$

So $x = \frac{1}{2}$ is a root, and $2x - 1$ divides $P(x)$.

Step 3: divide.

$$\begin{array}{r|l}
 2x^3 + x^2 + x - 1 & 2x - 1 \\
 - 2x^3 + x^2 & \hline
 2x^2 + x & \\
 - 2x^2 + x & \\
 \hline
 2x - 1 & \\
 - 2x + 1 & \\
 \hline
 0 &
 \end{array}$$

Hence

$$2x^3 + x^2 + x - 1 = (2x - 1)(x^2 + x + 1).$$

Step 4: solve the quadratic factor. For $x^2 + x + 1 = 0$ the discriminant is $\Delta = 1 - 4 = -3 < 0$, so there are no real roots (the complex roots are $x = \frac{-1 \pm i\sqrt{3}}{2}$).

Conclusion. The only real solution is

$$\boxed{x = \frac{1}{2}}.$$

□

2 Straight lines in the plane

Useful facts.

- The line through (x_1, y_1) and (x_2, y_2) with $x_1 \neq x_2$ has slope $m = \frac{y_2 - y_1}{x_2 - x_1}$ and equation $y - y_1 = m(x - x_1)$.
- If $x_1 = x_2$ the line is vertical, with equation $x = x_1$ (the slope is not defined).
- Two non-vertical lines with slopes m_1, m_2 are parallel iff $m_1 = m_2$, and perpendicular iff $m_1 m_2 = -1$, i.e. $m_2 = -\frac{1}{m_1}$.

Exercise 3.

Find the line through $(1, 1)$ and $(3, 1)$.

Solution. **Step 1: slope.** Since $x_1 = 1 \neq 3 = x_2$,

$$m = \frac{1 - 1}{3 - 1} = \frac{0}{2} = 0.$$

Step 2: equation. $y - 1 = 0 \cdot (x - 1)$, i.e.

$$\boxed{y = 1},$$

a horizontal line. **Check:** both points have y -coordinate 1. □

Exercise 4.

Find the line through $(3, 0)$ and $(3, 1)$.

Solution. **Step 1: slope?** Here $x_1 = x_2 = 3$, so the formula would give $m = \frac{1 - 0}{3 - 3} = \frac{1}{0}$, which is not defined: the line is *vertical*.

Step 2: equation. All points of the line have the same x -coordinate, so

$$\boxed{x = 3}.$$

Note that this line *cannot* be written in the form $y = mx + q$. **Check:** both points have x -coordinate 3. □

Exercise 5.

Find the line through $(5, 2)$ and $(6, 2)$.

Solution. **Step 1: slope.**

$$m = \frac{2 - 2}{6 - 5} = \frac{0}{1} = 0.$$

Step 2: equation. $y - 2 = 0 \cdot (x - 5)$, i.e.

$$\boxed{y = 2}.$$

Check: both points have y -coordinate 2. □

Exercise 6.

Find the line r such that $r \perp \{y = 2x - 1\}$ and $(1, 0) \in r$.

Solution. **Step 1: slope of the given line.** $y = 2x - 1$ has slope $m_1 = 2$.

Step 2: slope of r . Perpendicularity gives $m_1 m = -1$, so

$$m = -\frac{1}{2}.$$

Step 3: impose the passage through $(1, 0)$.

$$y - 0 = -\frac{1}{2}(x - 1) \implies \boxed{y = -\frac{1}{2}x + \frac{1}{2}},$$

or, in implicit form, $x + 2y - 1 = 0$.

Check: for $x = 1$ we get $y = -\frac{1}{2} + \frac{1}{2} = 0$; and $2 \cdot (-\frac{1}{2}) = -1$. □

Exercise 7.

Find the line r such that $r \parallel \{y = 3x + 2\}$ and $(0, 0) \in r$.

Solution. **Step 1: slope of the given line.** $y = 3x + 2$ has slope 3.

Step 2: slope of r . Parallel lines have the same slope, so $m = 3$ and $r : y = 3x + q$ for some q .

Step 3: impose the passage through $(0, 0)$.

$$0 = 3 \cdot 0 + q \implies q = 0.$$

Hence

$$\boxed{y = 3x}.$$

Check: $(0, 0)$ satisfies $y = 3x$, and the slope is 3 as required. □