

Classwork

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Exercise 1

Find the common solutions of the system

$$\begin{cases} 3x^3 - 3x + 1 = 0 \\ 2x^3 - 1 = 0 \end{cases}$$

Solution

A real number x_0 solves both equations if and only if it is a common root of $f(x) = 3x^3 - 3x + 1$ and $g(x) = 2x^3 - 1$, i.e. if and only if it is a root of $\gcd(f, g)$. We compute the gcd with the Euclidean algorithm.

First division: f divided by g .

$$\begin{array}{r|l} 3x^3 - 3x + 1 & 2x^3 - 1 \\ - 3x^3 & + \frac{3}{2} \\ \hline & -3x + \frac{5}{2} \end{array}$$

So

$$3x^3 - 3x + 1 = \frac{3}{2}(2x^3 - 1) + (-3x + \frac{5}{2}).$$

Since a gcd is defined up to a non-zero constant, we may replace the remainder $-3x + \frac{5}{2}$ by $-2(-3x + \frac{5}{2}) = 6x - 5$.

Second division: g divided by $6x - 5$.

$$\begin{array}{r|l} 2x^3 & -1 \quad 6x - 5 \\ - 2x^3 + \frac{5}{3}x^2 & \frac{1}{3}x^2 + \frac{5}{18}x + \frac{25}{108} \\ \hline & \frac{5}{3}x^2 \\ & - \frac{5}{3}x^2 + \frac{25}{18}x \\ & \hline & \frac{25}{18}x - 1 \\ & - \frac{25}{18}x + \frac{125}{108} \\ & \hline & \frac{17}{108} \end{array}$$

So

$$2x^3 - 1 = (6x - 5) \left(\frac{1}{3}x^2 + \frac{5}{18}x + \frac{25}{108} \right) + \frac{17}{108}.$$

The remainder $\frac{17}{108}$ is a non-zero constant, hence $\gcd(f, g) = 1$.

Conclusion. The two polynomials have no common root: the system has *no solutions*.

Check. The only candidate surviving the first step is the root of the remainder, $x = \frac{5}{6}$. Indeed $g(\frac{5}{6}) = 2 \cdot \frac{125}{216} - 1 = \frac{17}{108} \neq 0$, so $x = \frac{5}{6}$ is not a solution.

Exercise 2

Find the line r such that $(3, 2) \in r$ and $(4, 2) \in r$.

Solution

The two points have the same y -coordinate, so the slope is

$$m = \frac{2-2}{4-3} = 0.$$

The line is horizontal:

$$r : y = 2.$$

Exercise 3

Find the line r such that $(0, 0) \in r$ and $(0, 1) \in r$.

Solution

The two points have the same x -coordinate, so the slope $\frac{1-0}{0-0}$ is not defined: the line is vertical. All its points have $x = 0$, hence

$$r : x = 0$$

(the y -axis). Note that this line cannot be written in the form $y = mx + q$.

Exercise 4

Find the line r such that $r \perp (y = x)$ and $(3, 2) \in r$.

Solution

The line $y = x$ has slope 1. Two non-vertical lines with slopes m_1, m_2 are perpendicular if and only if $m_1 m_2 = -1$, so r has slope

$$m = -\frac{1}{1} = -1.$$

Imposing the passage through $(3, 2)$:

$$y - 2 = -1(x - 3) \implies r : y = -x + 5.$$

Check: $-3 + 5 = 2$. ✓

Exercise 5

Find the line r such that

$$\begin{cases} r \parallel (y = 2x + 4) \\ r \perp (y = x) \end{cases}$$

Solution

- $r \parallel (y = 2x + 4)$ means that r has the same slope as $y = 2x + 4$, that is $m = 2$.
- $r \perp (y = x)$ means, as in Exercise 4, that r has slope $m = -1$.

A line cannot have two different slopes at the same time ($2 \neq -1$), so the two conditions are incompatible: *there is no such line*.

Geometric interpretation. The lines $y = 2x + 4$ and $y = x$ are not perpendicular to each other ($2 \cdot 1 \neq -1$); a line parallel to the first and perpendicular to the second would force them to be perpendicular.