

FCS MATHo Thirs Lesson

Polynomial Equations: Ruffini's Theorem and Rational Roots

(II)

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Exercises

Solve the following equations in $\mathbb{R}$.

1. $x^3 + 2x^2 - 2x - 3 = 0$
2. $2x^3 - 3x^2 - 4x - 1 = 0$
3. $3x^3 + 2x^2 + 2x - 1 = 0$
4. $x^4 - x^3 - 4x^2 - 5x - 3 = 0$
5. $2x^4 + 3x^3 - 2x^2 - 2x + 1 = 0$
6. $x^4 + x^3 - 3x^2 - x + 2 = 0$

Solutions

Exercise 1: $x^3 + 2x^2 - 2x - 3 = 0$

Let $p(x) = x^3 + 2x^2 - 2x - 3$. It is monic, so the candidates are the divisors of 3:

$$\pm 1, \pm 3.$$

We test them:

$$p(1) = 1 + 2 - 2 - 3 = -2, \quad p(-1) = -1 + 2 + 2 - 3 = 0.$$

So $x = -1$ is a root. We divide by $x + 1$:

$$\begin{array}{r|l}
 x^3 & +2x^2 & -2x & -3 & x+1 \\
 -x^3 & -x^2 & & & x^2+x-3 \\
 \hline
 // & x^2 & -2x & -3 & \\
 & -x^2 & -x & & \\
 \hline
 & // & -3x & -3 & \\
 & & +3x & +3 & \\
 \hline
 & & // & 0 &
 \end{array}$$

$$\implies p(x) = (x+1)(x^2+x-3).$$

We solve $x^2 + x - 3 = 0$: $\Delta = 1 + 12 = 13$, so $x = \frac{-1 \pm \sqrt{13}}{2}$ (irrational, which is why they were not among the candidates). Hence the solutions are

$$x = -1, \quad x = \frac{-1 - \sqrt{13}}{2}, \quad x = \frac{-1 + \sqrt{13}}{2}.$$

Exercise 2: $2x^3 - 3x^2 - 4x - 1 = 0$

Let $p(x) = 2x^3 - 3x^2 - 4x - 1$. Here $a_0 = -1$ and $a_3 = 2$, so $r = 1$ and $s \in \{1, 2\}$, and the candidates are

$$\pm 1, \pm \frac{1}{2}.$$

We test them:

$$\begin{aligned} p(1) &= 2 - 3 - 4 - 1 = -6, & p(-1) &= -2 - 3 + 4 - 1 = -2, \\ p\left(\frac{1}{2}\right) &= \frac{1}{4} - \frac{3}{4} - 2 - 1 = -\frac{7}{2}, & p\left(-\frac{1}{2}\right) &= -\frac{1}{4} - \frac{3}{4} + 2 - 1 = 0. \end{aligned}$$

So $x = -\frac{1}{2}$ is a root. We divide by $2x + 1$:

$$\begin{array}{r|l} 2x^3 & -3x^2 & -4x & -1 & 2x+1 \\ -2x^3 & & & & \\ \hline & -4x^2 & -4x & -1 & \\ & +4x^2 & +2x & & \\ \hline & & -2x & -1 & \\ & & +2x & +1 & \\ \hline & & & 0 & \end{array}$$

$$\implies p(x) = (2x + 1)(x^2 - 2x - 1).$$

We solve $x^2 - 2x - 1 = 0$: $\frac{\Delta}{4} = 1 + 1 = 2$, so $x = 1 \pm \sqrt{2}$. Hence the solutions are

$$x = -\frac{1}{2}, \quad x = 1 - \sqrt{2}, \quad x = 1 + \sqrt{2}.$$

Exercise 3: $3x^3 + 2x^2 + 2x - 1 = 0$

Let $p(x) = 3x^3 + 2x^2 + 2x - 1$. Here $a_0 = -1$ and $a_3 = 3$, so $r = 1$ and $s \in \{1, 3\}$, and the candidates are

$$\pm 1, \pm \frac{1}{3}.$$

We test them:

$$\begin{aligned} p(1) &= 3 + 2 + 2 - 1 = 6, & p(-1) &= -3 + 2 - 2 - 1 = -4, \\ p\left(\frac{1}{3}\right) &= \frac{1}{9} + \frac{2}{9} + \frac{2}{3} - 1 = \frac{1 + 2 + 6 - 9}{9} = 0. \end{aligned}$$

So $x = \frac{1}{3}$ is a root. We divide by $3x - 1$:

$$\begin{array}{r|l} 3x^3 & +2x^2 & +2x & -1 & 3x-1 \\ -3x^3 & & & & \\ \hline & +2x^2 & +2x & -1 & \\ & -2x^2 & +2x & & \\ \hline & & +2x & -1 & \\ & & -2x & +1 & \\ \hline & & & 0 & \end{array}$$

$$\implies p(x) = (3x - 1)(x^2 + x + 1).$$

The equation $x^2 + x + 1 = 0$ has $\Delta = 1 - 4 = -3 < 0$, so it has no real solutions. Hence the only real solution is

$$x = \frac{1}{3}.$$

Exercise 4: $x^4 - x^3 - 4x^2 - 5x - 3 = 0$

Let $p(x) = x^4 - x^3 - 4x^2 - 5x - 3$. It is monic, so the candidates are the divisors of 3:

$$\pm 1, \pm 3.$$

We test them:

$$p(1) = 1 - 1 - 4 - 5 - 3 = -12, \quad p(-1) = 1 + 1 - 4 + 5 - 3 = 0.$$

So $x = -1$ is a root. We divide by $x + 1$:

$$\begin{array}{r|l} \begin{array}{rrrrr} x^4 & -x^3 & -4x^2 & -5x & -3 \\ -x^4 & -x^3 & & & \\ \hline & // & -2x^3 & -4x^2 & -5x & -3 \\ & & +2x^3 & +2x^2 & & \\ \hline & & // & -2x^2 & -5x & -3 \\ & & & +2x^2 & +2x & \\ \hline & & & // & -3x & -3 \\ & & & & +3x & +3 \\ \hline & & & & // & 0 \end{array} & \begin{array}{l} x + 1 \\ \hline x^3 - 2x^2 - 2x - 3 \end{array} \end{array}$$

$$\implies p(x) = (x + 1)q(x), \quad q(x) = x^3 - 2x^2 - 2x - 3.$$

The candidates for q are still $\pm 1, \pm 3$; 1 is excluded since it is not a root of p . We test:

$$q(-1) = -1 - 2 + 2 - 3 = -4, \quad q(3) = 27 - 18 - 6 - 3 = 0.$$

So $x = 3$ is a root of q . We divide by $x - 3$:

$$\begin{array}{r|l} \begin{array}{rrrr} x^3 & -2x^2 & -2x & -3 \\ -x^3 & +3x^2 & & \\ \hline & // & x^2 & -2x & -3 \\ & & -x^2 & +3x & \\ \hline & & // & x & -3 \\ & & & -x & +3 \\ \hline & & & // & 0 \end{array} & \begin{array}{l} x - 3 \\ \hline x^2 + x + 1 \end{array} \end{array}$$

$$\implies q(x) = (x - 3)(x^2 + x + 1).$$

The equation $x^2 + x + 1 = 0$ has $\Delta = -3 < 0$, so it has no real solutions. Hence

$$p(x) = (x + 1)(x - 3)(x^2 + x + 1),$$

and the real solutions are

$$x = -1, \quad x = 3.$$

Exercise 5: $2x^4 + 3x^3 - 2x^2 - 2x + 1 = 0$

Let $p(x) = 2x^4 + 3x^3 - 2x^2 - 2x + 1$. Here $a_0 = 1$ and $a_4 = 2$, so $r = 1$ and $s \in \{1, 2\}$, and the candidates are

$$\pm 1, \pm \frac{1}{2}.$$

We test them:

$$p(1) = 2 + 3 - 2 - 2 + 1 = 2, \quad p(-1) = 2 - 3 - 2 + 2 + 1 = 0.$$

So $x = -1$ is a root. We divide by $x + 1$:

$$\begin{array}{r|l} \begin{array}{rrrrr} 2x^4 & +3x^3 & -2x^2 & -2x & +1 \\ -2x^4 & -2x^3 & & & \\ \hline & // & x^3 & -2x^2 & -2x & +1 \\ & & -x^3 & -x^2 & & \\ & & \hline & & // & -3x^2 & -2x & +1 \\ & & & +3x^2 & +3x & \\ & & & \hline & & & // & x & +1 \\ & & & & -x & -1 \\ & & & & \hline & & & & // & 0 \end{array} & \begin{array}{l} x+1 \\ \hline 2x^3 + x^2 - 3x + 1 \end{array} \end{array}$$

$$\implies p(x) = (x+1)q(x), \quad q(x) = 2x^3 + x^2 - 3x + 1.$$

The candidates for q are $\pm 1, \pm \frac{1}{2}$; 1 is excluded, while -1 must be tested again since it could be a double root:

$$q(-1) = -2 + 1 + 3 + 1 = 3, \quad q\left(\frac{1}{2}\right) = \frac{1}{4} + \frac{1}{4} - \frac{3}{2} + 1 = 0.$$

So $x = \frac{1}{2}$ is a root of q . We divide by $2x - 1$:

$$\begin{array}{r|l} \begin{array}{rrrr} 2x^3 & +x^2 & -3x & +1 \\ -2x^3 & +x^2 & & \\ \hline & // & 2x^2 & -3x & +1 \\ & & -2x^2 & +x & \\ & & \hline & & // & -2x & +1 \\ & & & +2x & -1 \\ & & & \hline & & & // & 0 \end{array} & \begin{array}{l} 2x-1 \\ \hline x^2 + x - 1 \end{array} \end{array}$$

$$\implies q(x) = (2x-1)(x^2+x-1).$$

We solve $x^2 + x - 1 = 0$: $\Delta = 1 + 4 = 5$, so $x = \frac{-1 \pm \sqrt{5}}{2}$. Hence

$$p(x) = (x+1)(2x-1)(x^2+x-1),$$

and the solutions are

$$x = -1, \quad x = \frac{1}{2}, \quad x = \frac{-1 - \sqrt{5}}{2}, \quad x = \frac{-1 + \sqrt{5}}{2}.$$

Exercise 6: $x^4 + x^3 - 3x^2 - x + 2 = 0$

Let $p(x) = x^4 + x^3 - 3x^2 - x + 2$. It is monic, so the candidates are the divisors of 2:

$$\pm 1, \pm 2.$$

We have $p(1) = 1 + 1 - 3 - 1 + 2 = 0$, so $x = 1$ is a root. We divide by $x - 1$:

$$\begin{array}{r|l}
 \begin{array}{rrrrr}
 x^4 & +x^3 & -3x^2 & -x & +2 \\
 -x^4 & +x^3 & & & \\
 \hline
 // & 2x^3 & -3x^2 & -x & +2 \\
 & -2x^3 & +2x^2 & & \\
 \hline
 & // & -x^2 & -x & +2 \\
 & & +x^2 & -x & \\
 & & \hline
 & & // & -2x & +2 \\
 & & & +2x & -2 \\
 & & & \hline
 & & & // & 0
 \end{array}
 &
 \begin{array}{l}
 x-1 \\
 \hline
 x^3 + 2x^2 - x - 2
 \end{array}
 \end{array}$$

$$\implies p(x) = (x-1)q(x), \quad q(x) = x^3 + 2x^2 - x - 2.$$

The candidates for q are again $\pm 1, \pm 2$, and 1 must be tested again since it could be a double root: $q(1) = 1 + 2 - 1 - 2 = 0$. We divide by $x - 1$:

$$\begin{array}{r|l}
 \begin{array}{rrrr}
 x^3 & +2x^2 & -x & -2 \\
 -x^3 & +x^2 & & \\
 \hline
 // & 3x^2 & -x & -2 \\
 & -3x^2 & +3x & \\
 \hline
 & // & 2x & -2 \\
 & & -2x & +2 \\
 & & \hline
 & & // & 0
 \end{array}
 &
 \begin{array}{l}
 x-1 \\
 \hline
 x^2 + 3x + 2
 \end{array}
 \end{array}$$

$$\implies q(x) = (x-1)(x^2 + 3x + 2).$$

We solve $x^2 + 3x + 2 = 0$: $\Delta = 9 - 8 = 1$, so $x = \frac{-3 \pm 1}{2}$, that is, $x = -1$ and $x = -2$. Hence

$$p(x) = (x-1)^2(x+1)(x+2),$$

and the solutions are

$$x = 1 \text{ (double)}, \quad x = -1, \quad x = -2.$$