

FCS MATH0 Third Lesson

Polynomial Equations: Ruffini's Theorem and Rational Roots

Caboara

Recall

Theorem (Ruffini). *Let $p(x)$ be a polynomial and a a number. The remainder of the division of $p(x)$ by $x - a$ is $p(a)$. In particular,*

$$p(a) = 0 \iff (x - a) \text{ divides } p(x).$$

Proof. Dividing $p(x)$ by $x - a$ we get $p(x) = (x - a)q(x) + r$, with r a constant since $\deg r < \deg(x - a) = 1$. Substituting $x = a$ gives $p(a) = 0 \cdot q(a) + r = r$. $\square$

So every root a of p lowers the degree: $p(x) = (x - a)q(x)$ with $\deg q = \deg p - 1$, and the remaining roots of p are the roots of q . The quotient $q(x)$ is computed by polynomial long division.

Proposition (Rational roots). *Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ be a polynomial with integer coefficients, $a_n \neq 0$, $a_0 \neq 0$. If $\frac{r}{s}$, with r, s coprime integers and $s > 0$, is a root of p , then*

$$r \text{ divides } a_0 \quad \text{and} \quad s \text{ divides } a_n.$$

Proof. Multiplying $p(r/s) = 0$ by s^n gives

$$a_n r^n + a_{n-1} r^{n-1} s + \dots + a_1 r s^{n-1} + a_0 s^n = 0.$$

All terms except $a_0 s^n$ are multiples of r , so $r \mid a_0 s^n$; since $\gcd(r, s) = 1$, $r \mid a_0$. Similarly, all terms except $a_n r^n$ are multiples of s , so $s \mid a_n r^n$ and hence $s \mid a_n$. $\square$

Hence the only *candidate* rational roots are the fractions $\pm \frac{r}{s}$ with r a positive divisor of a_0 and s a positive divisor of a_n . If p is monic ($a_n = 1$), the candidates are just the divisors of a_0 , which are integers.

After finding a root a and dividing, the candidates for the quotient q are among those for p , and the candidates already rejected for p need not be tested again (except a itself, which may be a multiple root).

If the root is a fraction $a = \frac{r}{s}$, then $x - \frac{r}{s}$ divides $p(x)$, and so does $sx - r = s \left(x - \frac{r}{s} \right)$. It is convenient to divide by $sx - r$: in this way the quotient still has integer coefficients and no fractions appear in the computation.

Exercises

Solve the following equations in $\mathbb{R}$.

1. $x^3 + x^2 - 5x + 3 = 0$

2. $x^3 - x^2 - x - 2 = 0$

3. $2x^3 - 3x^2 - 3x + 2 = 0$
4. $x^4 - x^3 - x^2 - x - 2 = 0$
5. $2x^4 + x^3 - 7x^2 + 5x - 1 = 0$
6. $3x^4 + x^3 - 3x - 1 = 0$

Solutions

Exercise 1: $x^3 + x^2 - 5x + 3 = 0$

Let $p(x) = x^3 + x^2 - 5x + 3$. It is monic, so the candidates are the divisors of 3:

$$\pm 1, \pm 3.$$

We have $p(1) = 1 + 1 - 5 + 3 = 0$, so $x = 1$ is a root. We divide by $x - 1$:

$$\begin{array}{r|l}
 \begin{array}{rrrr}
 x^3 & +x^2 & -5x & +3 \\
 -x^3 & +x^2 & & \\
 \hline
 // & 2x^2 & -5x & +3 \\
 & -2x^2 & +2x & \\
 \hline
 & // & -3x & +3 \\
 & & +3x & -3 \\
 & & \hline
 & // & 0 &
 \end{array}
 &
 \begin{array}{l}
 x - 1 \\
 \hline
 x^2 + 2x - 3
 \end{array}
 \end{array}$$

$$\implies p(x) = (x - 1)(x^2 + 2x - 3).$$

We solve $x^2 + 2x - 3 = 0$: $\frac{\Delta}{4} = 1 + 3 = 4$, so $x = -1 \pm 2$, that is, $x = 1$ and $x = -3$. Hence

$$p(x) = (x - 1)^2(x + 3),$$

and the solutions are

$$x = 1 \text{ (double)}, \quad x = -3.$$

Exercise 2: $x^3 - x^2 - x - 2 = 0$

Let $p(x) = x^3 - x^2 - x - 2$. It is monic, so the candidates are the divisors of 2:

$$\pm 1, \pm 2.$$

We test them:

$$p(1) = 1 - 1 - 1 - 2 = -3, \quad p(-1) = -1 - 1 + 1 - 2 = -3, \quad p(2) = 8 - 4 - 2 - 2 = 0.$$

So $x = 2$ is a root. We divide by $x - 2$:

$$\begin{array}{r|l}
 \begin{array}{rrrr}
 x^3 & -x^2 & -x & -2 \\
 -x^3 & +2x^2 & & \\
 \hline
 // & x^2 & -x & -2 \\
 & -x^2 & +2x & \\
 \hline
 & // & x & -2 \\
 & & -x & +2 \\
 & & \hline
 & // & 0 &
 \end{array}
 &
 \begin{array}{l}
 x - 2 \\
 \hline
 x^2 + x + 1
 \end{array}
 \end{array}$$

$$\implies p(x) = (x-2)(x^2+x+1).$$

The equation $x^2 + x + 1 = 0$ has $\Delta = 1 - 4 = -3 < 0$, so it has no real solutions. Hence the only real solution is

$$x = 2.$$

Exercise 3: $2x^3 - 3x^2 - 3x + 2 = 0$

Let $p(x) = 2x^3 - 3x^2 - 3x + 2$. Here $a_0 = 2$ and $a_3 = 2$, so $r \in \{1, 2\}$ and $s \in \{1, 2\}$, and the candidates are

$$\pm 1, \pm 2, \pm \frac{1}{2}.$$

We test them:

$$p(1) = 2 - 3 - 3 + 2 = -2, \quad p(-1) = -2 - 3 + 3 + 2 = 0.$$

So $x = -1$ is a root. We divide by $x + 1$:

$$\begin{array}{r|l} 2x^3 & -3x^2 & -3x & +2 & x+1 \\ -2x^3 & -2x^2 & & & 2x^2 - 5x + 2 \\ \hline & // & -5x^2 & -3x & +2 \\ & & +5x^2 & +5x & \\ \hline & & // & 2x & +2 \\ & & & -2x & -2 \\ \hline & & & // & 0 \end{array}$$

$$\implies p(x) = (x+1)(2x^2 - 5x + 2).$$

We solve $2x^2 - 5x + 2 = 0$: $\Delta = 25 - 16 = 9$, so $x = \frac{5 \pm 3}{4}$, that is, $x = 2$ and $x = \frac{1}{2}$. Hence

$$p(x) = (x+1)(x-2)(2x-1),$$

and the solutions are

$$x = -1, \quad x = \frac{1}{2}, \quad x = 2.$$

Exercise 4: $x^4 - x^3 - x^2 - x - 2 = 0$

Let $p(x) = x^4 - x^3 - x^2 - x - 2$. It is monic, so the candidates are the divisors of 2:

$$\pm 1, \pm 2.$$

We test them:

$$p(1) = 1 - 1 - 1 - 1 - 2 = -4, \quad p(-1) = 1 + 1 - 1 + 1 - 2 = 0.$$

So $x = -1$ is a root. We divide by $x + 1$:

$$\begin{array}{r|l}
 \begin{array}{rrrrr}
 x^4 & -x^3 & -x^2 & -x & -2 \\
 -x^4 & -x^3 & & & \\
 \hline
 // & -2x^3 & -x^2 & -x & -2 \\
 & +2x^3 & +2x^2 & & \\
 \hline
 & // & x^2 & -x & -2 \\
 & & -x^2 & -x & \\
 & & \hline
 & & // & -2x & -2 \\
 & & & +2x & +2 \\
 & & & \hline
 & & & // & 0
 \end{array} &
 \begin{array}{l}
 x + 1 \\
 \hline
 x^3 - 2x^2 + x - 2
 \end{array}
 \end{array}$$

$$\implies p(x) = (x + 1)q(x), \quad q(x) = x^3 - 2x^2 + x - 2.$$

The candidates for q are still $\pm 1, \pm 2$; 1 is excluded since it is not a root of p . We test:

$$q(-1) = -1 - 2 - 1 - 2 = -6, \quad q(2) = 8 - 8 + 2 - 2 = 0.$$

So $x = 2$ is a root of q . We divide by $x - 2$:

$$\begin{array}{r|l}
 \begin{array}{rrrr}
 x^3 & -2x^2 & +x & -2 \\
 -x^3 & +2x^2 & & \\
 \hline
 // & // & x & -2 \\
 & & -x & +2 \\
 & & \hline
 & & // & 0
 \end{array} &
 \begin{array}{l}
 x - 2 \\
 \hline
 x^2 + 1
 \end{array}
 \end{array}$$

$$\implies q(x) = (x - 2)(x^2 + 1).$$

The equation $x^2 + 1 = 0$ has no real solutions. Hence

$$p(x) = (x + 1)(x - 2)(x^2 + 1),$$

and the real solutions are

$$x = -1, \quad x = 2.$$

Exercise 5: $2x^4 + x^3 - 7x^2 + 5x - 1 = 0$

Let $p(x) = 2x^4 + x^3 - 7x^2 + 5x - 1$. Here $a_0 = -1$ and $a_4 = 2$, so $r = 1$ and $s \in \{1, 2\}$, and the candidates are

$$\pm 1, \pm \frac{1}{2}.$$

We have $p(1) = 2 + 1 - 7 + 5 - 1 = 0$, so $x = 1$ is a root. We divide by $x - 1$:

$$\begin{array}{r|l}
 \begin{array}{rrrrr}
 2x^4 & +x^3 & -7x^2 & +5x & -1 \\
 -2x^4 & +2x^3 & & & \\
 \hline
 // & 3x^3 & -7x^2 & +5x & -1 \\
 & -3x^3 & +3x^2 & & \\
 & \hline
 & // & -4x^2 & +5x & -1 \\
 & & +4x^2 & -4x & \\
 & & \hline
 & & // & x & -1 \\
 & & & -x & +1 \\
 & & & \hline
 & & & // & 0
 \end{array} &
 \begin{array}{l}
 x - 1 \\
 \hline
 2x^3 + 3x^2 - 4x + 1
 \end{array}
 \end{array}$$

$$\implies p(x) = (x-1)q(x), \quad q(x) = 2x^3 + 3x^2 - 4x + 1.$$

The candidates for q are again $\pm 1, \pm \frac{1}{2}$. We test:

$$q(1) = 2 + 3 - 4 + 1 = 2, \quad q(-1) = -2 + 3 + 4 + 1 = 6, \quad q\left(\frac{1}{2}\right) = \frac{1}{4} + \frac{3}{4} - 2 + 1 = 0.$$

So $x = \frac{1}{2}$ is a root of q . We divide by $2x - 1$:

$$\begin{array}{r|l} \begin{array}{rrrr} 2x^3 & +3x^2 & -4x & +1 \\ -2x^3 & +x^2 & & \\ \hline & // & 4x^2 & -4x & +1 \\ & & -4x^2 & +2x & \\ \hline & & // & -2x & +1 \\ & & & +2x & -1 \\ \hline & & & // & 0 \end{array} & \begin{array}{l} 2x - 1 \\ \hline x^2 + 2x - 1 \end{array} \end{array}$$

$$\implies q(x) = (2x-1)(x^2+2x-1).$$

We solve $x^2 + 2x - 1 = 0$: $\frac{\Delta}{4} = 1 + 1 = 2$, so $x = -1 \pm \sqrt{2}$ (irrational, which is why they were not among the candidates). Hence

$$p(x) = (x-1)(2x-1)(x^2+2x-1),$$

and the solutions are

$$x = 1, \quad x = \frac{1}{2}, \quad x = -1 - \sqrt{2}, \quad x = -1 + \sqrt{2}.$$

Exercise 6: $3x^4 + x^3 - 3x - 1 = 0$

Let $p(x) = 3x^4 + x^3 + 0x^2 - 3x - 1$. Here $a_0 = -1$ and $a_4 = 3$, so $r = 1$ and $s \in \{1, 3\}$, and the candidates are

$$\pm 1, \pm \frac{1}{3}.$$

We have $p(1) = 3 + 1 - 3 - 1 = 0$, so $x = 1$ is a root. We divide by $x - 1$ (note the zero coefficient of x^2 in the dividend):

$$\begin{array}{r|l} \begin{array}{rrrr} 3x^4 & +x^3 & +0x^2 & -3x & -1 \\ -3x^4 & +3x^3 & & & \\ \hline & // & 4x^3 & & -3x & -1 \\ & & -4x^3 & +4x^2 & & \\ \hline & & // & 4x^2 & -3x & -1 \\ & & & -4x^2 & +4x & \\ \hline & & & // & x & -1 \\ & & & & -x & +1 \\ \hline & & & & // & 0 \end{array} & \begin{array}{l} x - 1 \\ \hline 3x^3 + 4x^2 + 4x + 1 \end{array} \end{array}$$

$$\implies p(x) = (x-1)q(x), \quad q(x) = 3x^3 + 4x^2 + 4x + 1.$$

The candidates for q are again $\pm 1, \pm \frac{1}{3}$. Since all the coefficients of q are positive, $q(x) > 0$ for $x > 0$, so only the negative candidates need to be tested:

$$q(-1) = -3 + 4 - 4 + 1 = -2, \quad q\left(-\frac{1}{3}\right) = -\frac{1}{9} + \frac{4}{9} - \frac{4}{3} + 1 = \frac{-1 + 4 - 12 + 9}{9} = 0.$$

So $x = -\frac{1}{3}$ is a root of q . We divide by $3x + 1$:

$$\begin{array}{r|l} \begin{array}{rrrr} 3x^3 & +4x^2 & +4x & +1 \\ -3x^3 & -x^2 & & \\ \hline // & 3x^2 & +4x & +1 \\ & -3x^2 & -x & \\ \hline & // & 3x & +1 \\ & & -3x & -1 \\ & & \hline & & // & 0 \end{array} & \begin{array}{l} 3x + 1 \\ \hline x^2 + x + 1 \end{array} \end{array}$$

$$\implies q(x) = (3x + 1)(x^2 + x + 1).$$

The equation $x^2 + x + 1 = 0$ has $\Delta = 1 - 4 = -3 < 0$, so it has no real solutions. Hence

$$p(x) = (x - 1)(3x + 1)(x^2 + x + 1),$$

and the real solutions are

$$x = 1, \quad x = -\frac{1}{3}.$$