

FCS MATH0 Third lesson

Classwork

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Exercises

1. Using the Euclidean algorithm, compute $\gcd(x^3 - 3x + 2, x^3 - x^2 - 3x + 6)$.
2. Solve the system

$$\begin{cases} x^3 + 3x^2 + x - 2 = 0 \\ x^3 + 2x^2 - 1 = 0. \end{cases}$$
3. Solve in $\mathbb{R}$ the equation $x^3 - 2x^2 - 2x + 1 = 0$.
4. Solve in $\mathbb{R}$ the equation $3x^3 - 4x^2 - 2x + 1 = 0$.
5. Solve in $\mathbb{R}$ the equation $2x^4 + x^3 - 5x^2 - 2x + 2 = 0$.

Solutions

Exercise 1

We compute the sequence of pairs $(f_{k+1}, g_{k+1}) = (g_k, \text{rem}(f_k : g_k))$, starting from

$$(f_0, g_0) = (x^3 - 3x + 2, x^3 - x^2 - 3x + 6).$$

Step 1. We divide f_0 by g_0 : $x^3 : x^3 = 1$.

$$\begin{array}{r|l} x^3 & +0x^2 & -3x & +2 & x^3 - x^2 - 3x + 6 \\ -x^3 & +x^2 & +3x & -6 & 1 \\ \hline // & x^2 & // & -4 & \end{array}$$

$$\Rightarrow (f_1, g_1) = (g_0, \text{rem}(f_0 : g_0)) = (x^3 - x^2 - 3x + 6, x^2 - 4).$$

Step 2. We divide f_1 by g_1 : $x^3 : x^2 = x$, then $-x^2 : x^2 = -1$.

$$\begin{array}{r|l} x^3 & -x^2 & -3x & +6 & x^2 - 4 \\ -x^3 & & +4x & & x - 1 \\ \hline // & -x^2 & +x & +6 & \\ & +x^2 & & -4 & \\ \hline // & & x & +2 & \end{array}$$

$$\Rightarrow (f_2, g_2) = (g_1, \text{rem}(f_1 : g_1)) = (x^2 - 4, x + 2).$$

Step 3. We divide f_2 by g_2 : $x^2 : x = x$, then $-2x : x = -2$.

$$\begin{array}{r|l} x^2 & +0x & -4 & x+2 \\ -x^2 & -2x & & x-2 \\ \hline // & -2x & -4 & \\ & +2x & +4 & \\ \hline // & 0 & & \end{array}$$

$$\implies (f_3, g_3) = (g_2, \text{rem}(f_2 : g_2)) = (x+2, 0).$$

Summary.

$$(x^3 - 3x + 2, x^3 - x^2 - 3x + 6) \rightarrow (x^3 - x^2 - 3x + 6, x^2 - 4) \rightarrow (x^2 - 4, x + 2) \rightarrow (x + 2, 0),$$

so, since $x + 2$ is already monic,

$$\gcd(x^3 - 3x + 2, x^3 - x^2 - 3x + 6) = x + 2.$$

Check. $x^3 - 3x + 2 = (x + 2)(x - 1)^2$ and $x^3 - x^2 - 3x + 6 = (x + 2)(x^2 - 3x + 3)$; since $1 - 3 + 3 = 1 \neq 0$, the cofactors have no common roots. $\checkmark$

Exercise 2

Let $p(x) = x^3 + 3x^2 + x - 2$ and $q(x) = x^3 + 2x^2 - 1$. A number α solves the system if and only if it is a common root of p and q , that is, if and only if $(x - \alpha)$ divides both p and q , that is, if and only if α is a root of $\gcd(p, q)$. We compute the GCD as a sequence of pairs $(f_{k+1}, g_{k+1}) = (g_k, \text{rem}(f_k : g_k))$ starting from $(f_0, g_0) = (p, q)$.

Step 1. We divide $f_0 = p$ by $g_0 = q$: $x^3 : x^3 = 1$.

$$\begin{array}{r|l} x^3 & +3x^2 & +x & -2 & x^3 + 2x^2 - 1 \\ -x^3 & -2x^2 & & +1 & 1 \\ \hline // & x^2 & +x & -1 & \end{array}$$

$$\implies (f_1, g_1) = (g_0, \text{rem}(f_0 : g_0)) = (x^3 + 2x^2 - 1, x^2 + x - 1).$$

Step 2. We divide f_1 by g_1 : $x^3 : x^2 = x$, then $x^2 : x^2 = 1$.

$$\begin{array}{r|l} x^3 & +2x^2 & +0x & -1 & x^2 + x - 1 \\ -x^3 & -x^2 & +x & & x + 1 \\ \hline // & x^2 & +x & -1 & \\ & -x^2 & -x & +1 & \\ \hline // & // & 0 & & \end{array}$$

$$\implies (f_2, g_2) = (g_1, \text{rem}(f_1 : g_1)) = (x^2 + x - 1, 0).$$

Summary.

$$(x^3 + 3x^2 + x - 2, x^3 + 2x^2 - 1) \rightarrow (x^3 + 2x^2 - 1, x^2 + x - 1) \rightarrow (x^2 + x - 1, 0),$$

hence

$$\gcd(p, q) = x^2 + x - 1.$$

The solutions of the system are the roots of $x^2 + x - 1$: $\Delta = 1 + 4 = 5$, so

$$x = \frac{-1 - \sqrt{5}}{2}, \quad x = \frac{-1 + \sqrt{5}}{2}.$$

Check. $p(x) = (x^2 + x - 1)(x + 2)$ and $q(x) = (x^2 + x - 1)(x + 1)$; the cofactors $x + 2$ and $x + 1$ have no common roots, so there are no other solutions.

Exercise 3: $x^3 - 2x^2 - 2x + 1 = 0$

Let $p(x) = x^3 - 2x^2 - 2x + 1$. It is monic and $a_0 = 1$, so the only candidate rational roots are ± 1 .

We test them:

$$p(1) = 1 - 2 - 2 + 1 = -2, \quad p(-1) = -1 - 2 + 2 + 1 = 0.$$

So $x = -1$ is a root. We divide by $x + 1$:

$$\begin{array}{r|l} x^3 & -2x^2 & -2x & +1 & x+1 \\ -x^3 & & & & \\ \hline // & -3x^2 & -2x & +1 & \\ & +3x^2 & +3x & & \\ \hline & // & x & +1 & \\ & & -x & -1 & \\ \hline & & // & 0 & \end{array}$$

$$\implies p(x) = (x + 1)(x^2 - 3x + 1).$$

We solve $x^2 - 3x + 1 = 0$: $\Delta = 9 - 4 = 5$, so $x = \frac{3 \pm \sqrt{5}}{2}$. Hence the solutions are

$$x = -1, \quad x = \frac{3 - \sqrt{5}}{2}, \quad x = \frac{3 + \sqrt{5}}{2}.$$

Exercise 4: $3x^3 - 4x^2 - 2x + 1 = 0$

Let $p(x) = 3x^3 - 4x^2 - 2x + 1$. Here $a_0 = 1$ and $a_3 = 3$, so $r = 1$ and $s \in \{1, 3\}$, and the candidates are

$$\pm 1, \pm \frac{1}{3}.$$

We test them:

$$p(1) = 3 - 4 - 2 + 1 = -2, \quad p(-1) = -3 - 4 + 2 + 1 = -4,$$

$$p\left(\frac{1}{3}\right) = \frac{1}{9} - \frac{4}{9} - \frac{2}{3} + 1 = \frac{1 - 4 - 6 + 9}{9} = 0.$$

So $x = \frac{1}{3}$ is a root, and $3x - 1$ divides $p(x)$. We divide by $3x - 1$:

$$\begin{array}{r|l} 3x^3 & -4x^2 & -2x & +1 & 3x-1 \\ -3x^3 & & & & \\ \hline // & -3x^2 & -2x & +1 & \\ & +3x^2 & -x & & \\ \hline & // & -3x & +1 & \\ & & +3x & -1 & \\ \hline & & // & 0 & \end{array}$$

$$\implies p(x) = (3x - 1)(x^2 - x - 1).$$

We solve $x^2 - x - 1 = 0$: $\Delta = 1 + 4 = 5$, so $x = \frac{1 \pm \sqrt{5}}{2}$. Hence the solutions are

$$x = \frac{1}{3}, \quad x = \frac{1 - \sqrt{5}}{2}, \quad x = \frac{1 + \sqrt{5}}{2}.$$

Exercise 5: $2x^4 + x^3 - 5x^2 - 2x + 2 = 0$

Let $p(x) = 2x^4 + x^3 - 5x^2 - 2x + 2$. Here $a_0 = 2$ and $a_4 = 2$, so $r \in \{1, 2\}$ and $s \in \{1, 2\}$, and the candidates are

$$\pm 1, \pm 2, \pm \frac{1}{2}.$$

We test them:

$$p(1) = 2 + 1 - 5 - 2 + 2 = -2, \quad p(-1) = 2 - 1 - 5 + 2 + 2 = 0.$$

So $x = -1$ is a root. We divide by $x + 1$:

$$\begin{array}{r|l} \begin{array}{rrrrr} 2x^4 & +x^3 & -5x^2 & -2x & +2 \\ -2x^4 & -2x^3 & & & \\ \hline & // & -x^3 & -5x^2 & -2x & +2 \\ & & +x^3 & +x^2 & & \\ & & \hline & & // & -4x^2 & -2x & +2 \\ & & & +4x^2 & +4x & \\ & & & \hline & & & // & 2x & +2 \\ & & & & -2x & -2 \\ & & & & \hline & & & & // & 0 \end{array} & \begin{array}{l} x+1 \\ \hline 2x^3 - x^2 - 4x + 2 \end{array} \end{array}$$

$$\implies p(x) = (x+1)q(x), \quad q(x) = 2x^3 - x^2 - 4x + 2.$$

The candidates for q are among those for p ; 1 is excluded, while -1 must be tested again since it could be a double root:

$$q(-1) = -2 - 1 + 4 + 2 = 3, \quad q(2) = 16 - 4 - 8 + 2 = 6, \quad q(-2) = -16 - 4 + 8 + 2 = -10,$$

$$q\left(\frac{1}{2}\right) = \frac{1}{4} - \frac{1}{4} - 2 + 2 = 0.$$

So $x = \frac{1}{2}$ is a root of q , and $2x - 1$ divides $q(x)$. We divide by $2x - 1$:

$$\begin{array}{r|l} \begin{array}{rrrr} 2x^3 & -x^2 & -4x & +2 \\ -2x^3 & +x^2 & & \\ \hline & // & // & -4x & +2 \\ & & & +4x & -2 \\ & & & \hline & & & // & 0 \end{array} & \begin{array}{l} 2x-1 \\ \hline x^2-2 \end{array} \end{array}$$

$$\implies q(x) = (2x-1)(x^2-2).$$

We solve $x^2 - 2 = 0$: $x = \pm\sqrt{2}$. Hence

$$p(x) = (x+1)(2x-1)(x^2-2),$$

and the solutions are

$$x = -1, \quad x = \frac{1}{2}, \quad x = -\sqrt{2}, \quad x = \sqrt{2}.$$