

# FCS MATH0 Third Lesson

## Exercise 3.2

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### Exercise 1

Using the Euclidean algorithm, compute the greatest common divisor of the following pairs of polynomials.

$$(a) \quad A(x) = x^3 - 2x^2 - 4x + 5, \quad B(x) = x^2 + x - 2;$$

$$(b) \quad A(x) = x^3 + x^2 - 3x - 3, \quad B(x) = x^3 - 2x - 1.$$

### Solution

$$(a) \quad (f_0, g_0) = (x^3 - 2x^2 - 4x + 5, x^2 + x - 2).$$

**Step 1.** We divide  $f_0$  by  $g_0$ :  $x^3 : x^2 = x$ , then  $-3x^2 : x^2 = -3$ .

$$\begin{array}{r|l} \begin{array}{rrrr} x^3 & -2x^2 & -4x & +5 \\ -x^3 & -x^2 & +2x & \\ \hline // & -3x^2 & -2x & +5 \\ & +3x^2 & +3x & -6 \\ \hline // & & x & -1 \end{array} & \begin{array}{l} x^2 + x - 2 \\ \hline x - 3 \end{array} \end{array}$$

$$\implies (f_1, g_1) = (g_0, \text{rem}(f_0 : g_0)) = (x^2 + x - 2, x - 1).$$

**Step 2.** We divide  $f_1$  by  $g_1$ :  $x^2 : x = x$ , then  $2x : x = 2$ .

$$\begin{array}{r|l} \begin{array}{rrr} x^2 & +x & -2 \\ -x^2 & +x & \\ \hline // & 2x & -2 \\ & -2x & +2 \\ \hline // & & 0 \end{array} & \begin{array}{l} x - 1 \\ \hline x + 2 \end{array} \end{array}$$

$$\implies (f_2, g_2) = (g_1, \text{rem}(f_1 : g_1)) = (x - 1, 0).$$

**Summary.**

$$(x^3 - 2x^2 - 4x + 5, x^2 + x - 2) \rightarrow (x^2 + x - 2, x - 1) \rightarrow (x - 1, 0),$$

so, since  $x - 1$  is already monic,

$$\gcd(A, B) = x - 1.$$

**Check.**  $A(x) = (x - 1)(x^2 - x - 5)$  and  $B(x) = (x - 1)(x + 2)$ ; since  $(-2)^2 - (-2) - 5 = 1 \neq 0$ , the cofactors have no common roots. ✓

(b)  $(f_0, g_0) = (x^3 + x^2 - 3x - 3, x^3 - 2x - 1)$ .

**Step 1.** We divide  $f_0$  by  $g_0$ :  $x^3 : x^3 = 1$ .

$$\begin{array}{rrrr|l} x^3 & +x^2 & -3x & -3 & x^3 - 2x - 1 \\ -x^3 & & +2x & +1 & 1 \\ \hline // & x^2 & -x & -2 & \end{array}$$

$$\implies (f_1, g_1) = (g_0, \text{rem}(f_0 : g_0)) = (x^3 - 2x - 1, x^2 - x - 2).$$

**Step 2.** We divide  $f_1$  by  $g_1$ :  $x^3 : x^2 = x$ , then  $x^2 : x^2 = 1$ .

$$\begin{array}{rrrr|l} x^3 & +0x^2 & -2x & -1 & x^2 - x - 2 \\ -x^3 & +x^2 & +2x & & x + 1 \\ \hline // & x^2 & // & -1 & \\ & -x^2 & +x & +2 & \\ \hline // & x & +1 & & \end{array}$$

$$\implies (f_2, g_2) = (g_1, \text{rem}(f_1 : g_1)) = (x^2 - x - 2, x + 1).$$

**Step 3.** We divide  $f_2$  by  $g_2$ :  $x^2 : x = x$ , then  $-2x : x = -2$ .

$$\begin{array}{rrr|l} x^2 & -x & -2 & x + 1 \\ -x^2 & -x & & x - 2 \\ \hline // & -2x & -2 & \\ & +2x & +2 & \\ \hline // & 0 & & \end{array}$$

$$\implies (f_3, g_3) = (g_2, \text{rem}(f_2 : g_2)) = (x + 1, 0).$$

**Summary.**

$$(x^3 + x^2 - 3x - 3, x^3 - 2x - 1) \rightarrow (x^3 - 2x - 1, x^2 - x - 2) \rightarrow (x^2 - x - 2, x + 1) \rightarrow (x + 1, 0),$$

so, since  $x + 1$  is already monic,

$$\gcd(A, B) = x + 1.$$

**Check.**  $A(x) = (x + 1)(x^2 - 3)$  and  $B(x) = (x + 1)(x^2 - x - 1)$ ; a common root of the cofactors would satisfy their difference  $x - 2 = 0$ , but  $2^2 - 3 = 1 \neq 0$ , so the cofactors have no common roots.  $\checkmark$

## Exercise 2

Solve the system

$$\begin{cases} x^3 - 2x - 4 = 0 \\ x^3 - x^2 - x - 2 = 0. \end{cases}$$

## Solution

Let  $p(x) = x^3 - 2x - 4$  and  $q(x) = x^3 - x^2 - x - 2$ . A number  $\alpha$  solves the system if and only if it is a common root of  $p$  and  $q$ , that is, if and only if  $(x - \alpha)$  divides both  $p$  and  $q$ , that is, if and only if  $\alpha$  is a root of  $\gcd(p, q)$ . So we compute the GCD with the Euclidean algorithm, as a sequence of pairs  $(f_{k+1}, g_{k+1}) = (g_k, \text{rem}(f_k : g_k))$  starting from  $(f_0, g_0) = (p, q)$ .

**Step 1.** We divide  $f_0 = p$  by  $g_0 = q$ :  $x^3 : x^3 = 1$ .

$$\begin{array}{rrrr|l} x^3 & +0x^2 & -2x & -4 & x^3 - x^2 - x - 2 \\ -x^3 & +x^2 & +x & +2 & 1 \\ \hline // & x^2 & -x & -2 & \end{array}$$

$$\Rightarrow (f_1, g_1) = (g_0, \text{rem}(f_0 : g_0)) = (x^3 - x^2 - x - 2, x^2 - x - 2).$$

**Step 2.** We divide  $f_1$  by  $g_1$ :  $x^3 : x^2 = x$ .

$$\begin{array}{rrrr|l} x^3 & -x^2 & -x & -2 & x^2 - x - 2 \\ -x^3 & +x^2 & +2x & & x \\ \hline // & // & x & -2 & \end{array}$$

$$\Rightarrow (f_2, g_2) = (g_1, \text{rem}(f_1 : g_1)) = (x^2 - x - 2, x - 2).$$

**Step 3.** We divide  $f_2$  by  $g_2$ :  $x^2 : x = x$ , then  $x : x = 1$ .

$$\begin{array}{rrr|l} x^2 & -x & -2 & x - 2 \\ -x^2 & +2x & & x + 1 \\ \hline // & x & -2 & \\ & -x & +2 & \\ & // & 0 & \end{array}$$

$$\Rightarrow (f_3, g_3) = (g_2, \text{rem}(f_2 : g_2)) = (x - 2, 0).$$

**Summary.**

$$(x^3 - 2x - 4, x^3 - x^2 - x - 2) \rightarrow (x^3 - x^2 - x - 2, x^2 - x - 2) \rightarrow (x^2 - x - 2, x - 2) \rightarrow (x - 2, 0).$$

Hence

$$\gcd(p, q) = x - 2,$$

and the only solution of the system is

$$x = 2.$$

**Check.**  $p(2) = 8 - 4 - 4 = 0$  and  $q(2) = 8 - 4 - 2 - 2 = 0$ . Moreover  $p(x) = (x - 2)(x^2 + 2x + 2)$  and  $q(x) = (x - 2)(x^2 + x + 1)$ ; the cofactors have no common roots (a common root would satisfy their difference  $x + 1 = 0$ , but  $(-1)^2 + (-1) + 1 = 1 \neq 0$ ), so there are no other solutions, not even complex ones.