

FCS MATH0 Thirs Lesson

Polynomials: Division, GCD and Systems

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Exercise 1

Perform the long division

$$(2x^4 - x^3 + 3x^2 - 5x + 1) : (x^3 - 2x + 1),$$

finding the quotient $Q(x)$ and the remainder $R(x)$.

Solution

We write the dividend and the divisor with all powers: $2x^4 - x^3 + 3x^2 - 5x + 1$ and $x^3 + 0x^2 - 2x + 1$.

- **Step 1.** $2x^4 : x^3 = 2x$. $2x(x^3 - 2x + 1) = 2x^4 - 4x^2 + 2x$; with changed signs:
 $-2x^4 + 4x^2 - 2x$.
 Add: $-x^3 + 7x^2 - 7x + 1$.
- **Step 2.** $-x^3 : x^3 = -1$. $-1 \cdot (x^3 - 2x + 1) = -x^3 + 2x - 1$; with changed signs:
 $x^3 - 2x + 1$.
 Add: $7x^2 - 9x + 2$.

The remainder has degree $2 < 3$, so we stop.

$$\begin{array}{r|l}
 \begin{array}{rrrrr}
 2x^4 & -x^3 & +3x^2 & -5x & +1 \\
 -2x^4 & & +4x^2 & -2x & \\
 \hline
 // & -x^3 & +7x^2 & -7x & +1 \\
 & +x^3 & & -2x & +1 \\
 \hline
 // & & 7x^2 & -9x & +2
 \end{array}
 &
 \begin{array}{l}
 x^3 - 2x + 1 \\
 \hline
 2x - 1
 \end{array}
 \end{array}$$

$$Q(x) = 2x - 1, \quad R(x) = 7x^2 - 9x + 2.$$

Check. $(x^3 - 2x + 1)(2x - 1) + 7x^2 - 9x + 2 = 2x^4 - x^3 - 4x^2 + 4x - 1 + 7x^2 - 9x + 2 = 2x^4 - x^3 + 3x^2 - 5x + 1$. ✓

Exercise 2

Using the Euclidean algorithm, compute the greatest common divisor of the following pairs of polynomials.

(a) $A(x) = 2x^3 + x^2 + 6x + 4$, $B(x) = x^3 + 3x + 1$;

(b) $A(x) = x^4 + 4x^3 + 7x^2 + 8x + 4$, $B(x) = x^3 + 3x^2 + 3x + 2$.

Solution

Recall. If $f = gq + r$ with $r = \text{rem}(f : g)$, then every common divisor of f and g divides r , and every common divisor of g and r divides f ; hence

$$\gcd(f, g) = \gcd(g, \text{rem}(f : g)).$$

Starting from $(f_0, g_0) = (A, B)$ we build the sequence of pairs

$$(f_{k+1}, g_{k+1}) = (g_k, \text{rem}(f_k : g_k)),$$

all with the same GCD, in which the degree of the second polynomial strictly decreases. We stop when we reach a pair $(f_n, 0)$: then $\gcd(A, B) = \gcd(f_n, 0)$ is f_n made monic.

(a) $(f_0, g_0) = (2x^3 + x^2 + 6x + 4, x^3 + 3x + 1).$

Step 1. We divide f_0 by g_0 : $2x^3 : x^3 = 2$.

$$\begin{array}{r|l} 2x^3 & +x^2 & +6x & +4 & x^3 + 3x + 1 \\ -2x^3 & & -6x & -2 & 2 \\ \hline // & x^2 & // & +2 & \end{array}$$

$$\Rightarrow (f_1, g_1) = (g_0, \text{rem}(f_0 : g_0)) = (x^3 + 3x + 1, x^2 + 2).$$

Step 2. We divide f_1 by g_1 : $x^3 : x^2 = x$.

$$\begin{array}{r|l} x^3 & +0x^2 & +3x & +1 & x^2 + 2 \\ -x^3 & & -2x & & x \\ \hline // & & x & +1 & \end{array}$$

$$\Rightarrow (f_2, g_2) = (g_1, \text{rem}(f_1 : g_1)) = (x^2 + 2, x + 1).$$

Step 3. We divide f_2 by g_2 : $x^2 : x = x$, then $-x : x = -1$.

$$\begin{array}{r|l} x^2 & +0x & +2 & x + 1 \\ -x^2 & -x & & x - 1 \\ \hline // & -x & +2 & \\ & +x & +1 & \\ \hline & // & 3 & \end{array}$$

$$\Rightarrow (f_3, g_3) = (g_2, \text{rem}(f_2 : g_2)) = (x + 1, 3).$$

Step 4. $g_3 = 3$ is a nonzero constant, so it divides f_3 exactly and

$$(f_4, g_4) = (g_3, \text{rem}(f_3 : g_3)) = (3, 0).$$

Summary.

$$\begin{aligned} (2x^3 + x^2 + 6x + 4, x^3 + 3x + 1) &\rightarrow (x^3 + 3x + 1, x^2 + 2) \rightarrow (x^2 + 2, x + 1) \\ &\rightarrow (x + 1, 3) \rightarrow (3, 0), \end{aligned}$$

so

$$\gcd(A, B) = 1,$$

that is, A and B are coprime.

(b) $(f_0, g_0) = (x^4 + 4x^3 + 7x^2 + 8x + 4, x^3 + 3x^2 + 3x + 2)$.

Step 1. We divide f_0 by g_0 : $x^4 : x^3 = x$, then $x^3 : x^3 = 1$.

$$\begin{array}{r|l} \begin{array}{rrrrr} x^4 & +4x^3 & +7x^2 & +8x & +4 \\ -x^4 & -3x^3 & -3x^2 & -2x & \\ \hline & // & x^3 & +4x^2 & +6x & +4 \\ & & -x^3 & -3x^2 & -3x & -2 \\ & & \hline & & // & x^2 & +3x & +2 \end{array} & \begin{array}{l} x^3 + 3x^2 + 3x + 2 \\ \hline x + 1 \end{array} \end{array}$$

$$\Rightarrow (f_1, g_1) = (g_0, \text{rem}(f_0 : g_0)) = (x^3 + 3x^2 + 3x + 2, x^2 + 3x + 2).$$

Step 2. We divide f_1 by g_1 : $x^3 : x^2 = x$.

$$\begin{array}{r|l} \begin{array}{rrrr} x^3 & +3x^2 & +3x & +2 \\ -x^3 & -3x^2 & -2x & \\ \hline & // & // & x & +2 \end{array} & \begin{array}{l} x^2 + 3x + 2 \\ \hline x \end{array} \end{array}$$

$$\Rightarrow (f_2, g_2) = (g_1, \text{rem}(f_1 : g_1)) = (x^2 + 3x + 2, x + 2).$$

Step 3. We divide f_2 by g_2 : $x^2 : x = x$, then $x : x = 1$.

$$\begin{array}{r|l} \begin{array}{rrr} x^2 & +3x & +2 \\ -x^2 & -2x & \\ \hline & // & x & +2 \\ & & -x & -2 \\ & & \hline & & // & 0 \end{array} & \begin{array}{l} x + 2 \\ \hline x + 1 \end{array} \end{array}$$

$$\Rightarrow (f_3, g_3) = (g_2, \text{rem}(f_2 : g_2)) = (x + 2, 0).$$

Summary.

$$\begin{aligned} (x^4 + 4x^3 + 7x^2 + 8x + 4, x^3 + 3x^2 + 3x + 2) &\rightarrow (x^3 + 3x^2 + 3x + 2, x^2 + 3x + 2) \\ &\rightarrow (x^2 + 3x + 2, x + 2) \rightarrow (x + 2, 0), \end{aligned}$$

so, since $x + 2$ is already monic,

$$\gcd(A, B) = x + 2.$$

Check. $A(x) = (x + 2)(x + 1)(x^2 + x + 2)$ and $B(x) = (x + 2)(x^2 + x + 1)$; the factors $(x + 1)(x^2 + x + 2)$ and $x^2 + x + 1$ have no common roots. ✓

Exercise 3

Solve the system

$$\begin{cases} x^4 - x^2 - 2 = 0 \\ x^4 - x^3 - x^2 + 2x - 2 = 0. \end{cases}$$

Solution

Let $p(x) = x^4 - x^2 - 2$ and $q(x) = x^4 - x^3 - x^2 + 2x - 2$. A number α solves the system if and only if it is a common root of p and q , that is, if and only if $(x - \alpha)$ divides both p and q , that is, if and only if α is a root of $\gcd(p, q)$. So we compute the GCD with the Euclidean algorithm, as a sequence of pairs $(f_{k+1}, g_{k+1}) = (g_k, \text{rem}(f_k : g_k))$ starting from $(f_0, g_0) = (p, q)$.

Step 1. We divide $f_0 = p$ by $g_0 = q$: $x^4 : x^4 = 1$.

$$\begin{array}{r|l} \begin{array}{rrrrr} x^4 & +0x^3 & -x^2 & +0x & -2 \\ -x^4 & +x^3 & +x^2 & -2x & +2 \\ \hline // & x^3 & // & -2x & // \end{array} & \begin{array}{l} x^4 - x^3 - x^2 + 2x - 2 \\ 1 \end{array} \end{array}$$

$$\implies (f_1, g_1) = (g_0, \text{rem}(f_0 : g_0)) = (x^4 - x^3 - x^2 + 2x - 2, x^3 - 2x).$$

Step 2. We divide f_1 by g_1 : $x^4 : x^3 = x$, then $-x^3 : x^3 = -1$.

$$\begin{array}{r|l} \begin{array}{rrrrr} x^4 & -x^3 & -x^2 & +2x & -2 \\ -x^4 & & +2x^2 & & \\ \hline // & -x^3 & +x^2 & +2x & -2 \\ & +x^3 & & -2x & \\ \hline // & x^2 & // & -2 & \end{array} & \begin{array}{l} x^3 - 2x \\ x - 1 \end{array} \end{array}$$

$$\implies (f_2, g_2) = (g_1, \text{rem}(f_1 : g_1)) = (x^3 - 2x, x^2 - 2).$$

Step 3. We divide f_2 by g_2 : $x^3 : x^2 = x$.

$$\begin{array}{r|l} \begin{array}{rrr} x^3 & +0x^2 & -2x \\ -x^3 & & +2x \\ \hline // & & 0 \end{array} & \begin{array}{l} x^2 - 2 \\ x \end{array} \end{array}$$

$$\implies (f_3, g_3) = (g_2, \text{rem}(f_2 : g_2)) = (x^2 - 2, 0).$$

Summary.

$$\begin{aligned} (x^4 - x^2 - 2, x^4 - x^3 - x^2 + 2x - 2) &\rightarrow (x^4 - x^3 - x^2 + 2x - 2, x^3 - 2x) \\ &\rightarrow (x^3 - 2x, x^2 - 2) \rightarrow (x^2 - 2, 0). \end{aligned}$$

Hence

$$\gcd(p, q) = x^2 - 2,$$

and the solutions of the system are the roots of $x^2 - 2$:

$$x = -\sqrt{2}, \quad x = \sqrt{2}.$$

Check. Indeed $p(x) = (x^2 - 2)(x^2 + 1)$ and $q(x) = (x^2 - 2)(x^2 - x + 1)$. The cofactors $x^2 + 1$ and $x^2 - x + 1$ have no common roots (a common root would satisfy their difference $x = 0$, which is a root of neither), so there are no other solutions, not even complex ones.