

FCS Math0 - Second Lesson

GCD by Prime Factorization

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Method

To compute $\gcd(a, b)$:

1. Factor a and b into primes by repeated division: divide by the smallest prime that divides the current quotient, and continue until the quotient is 1.
2. Write $a = \prod p^{\alpha_p}$ and $b = \prod p^{\beta_p}$ and keep only the primes that appear in *both* factorizations.
3. Then $\gcd(a, b) = \prod_p p^{\min(\alpha_p, \beta_p)}$.

In the division tables, the left column contains the current quotient and the right column the prime by which it is divided.

1 Two- and three-digit numbers

Example 1: $\gcd(84, 120)$

Step 1: factor both numbers by repeated division.

84		2	120		2
42		2	60		2
21		3	30		2
7		7	15		3
1			5		5
			1		

Hence

$$84 = 2^2 \cdot 3 \cdot 7, \quad 120 = 2^3 \cdot 3 \cdot 5.$$

Step 2: take the common primes with the smallest exponent.

- $p = 2$: exponents 2 and 3, so $\min(2, 3) = 2$;
- $p = 3$: exponents 1 and 1, so $\min(1, 1) = 1$.

The primes 5, 7 appear in only one factorization and do not contribute.

Step 3: multiply.

$$\gcd(84, 120) = 2^2 \cdot 3 = 4 \cdot 3 = 12.$$

Check: $84 = 12 \cdot 7$ and $120 = 12 \cdot 10$, with $\gcd(7, 10) = 1$.

Example 2: $\gcd(36, 60)$

Step 1: factor both numbers by repeated division.

$$\begin{array}{r|l} 36 & 2 \\ 18 & 2 \\ 9 & 3 \\ 3 & 3 \\ 1 & \end{array} \qquad \begin{array}{r|l} 60 & 2 \\ 30 & 2 \\ 15 & 3 \\ 5 & 5 \\ 1 & \end{array}$$

Hence

$$36 = 2^2 \cdot 3^2, \quad 60 = 2^2 \cdot 3 \cdot 5.$$

Step 2: take the common primes with the smallest exponent.

- $p = 2$: exponents 2 and 2, so $\min(2, 2) = 2$;
- $p = 3$: exponents 2 and 1, so $\min(2, 1) = 1$.

The prime 5 appears in only one factorization and does not contribute.

Step 3: multiply.

$$\gcd(36, 60) = 2^2 \cdot 3 = 4 \cdot 3 = 12.$$

Check: $36 = 12 \cdot 3$ and $60 = 12 \cdot 5$, with $\gcd(3, 5) = 1$.

Example 3: $\gcd(72, 96)$

Step 1: factor both numbers by repeated division.

$$\begin{array}{r|l} 72 & 2 \\ 36 & 2 \\ 18 & 2 \\ 9 & 3 \\ 3 & 3 \\ 1 & \end{array} \qquad \begin{array}{r|l} 96 & 2 \\ 48 & 2 \\ 24 & 2 \\ 12 & 2 \\ 6 & 2 \\ 3 & 3 \\ 1 & \end{array}$$

Hence

$$72 = 2^3 \cdot 3^2, \quad 96 = 2^5 \cdot 3.$$

Step 2: take the common primes with the smallest exponent.

- $p = 2$: exponents 3 and 5, so $\min(3, 5) = 3$;
- $p = 3$: exponents 2 and 1, so $\min(2, 1) = 1$.

Step 3: multiply.

$$\gcd(72, 96) = 2^3 \cdot 3 = 8 \cdot 3 = 24.$$

Check: $72 = 24 \cdot 3$ and $96 = 24 \cdot 4$, with $\gcd(3, 4) = 1$.

Example 4: $\gcd(90, 126)$

Step 1: factor both numbers by repeated division.

90	2	126	2
45	3	63	3
15	3	21	3
5	5	7	7
1		1	

Hence

$$90 = 2 \cdot 3^2 \cdot 5, \quad 126 = 2 \cdot 3^2 \cdot 7.$$

Step 2: take the common primes with the smallest exponent.

- $p = 2$: exponents 1 and 1, so $\min(1, 1) = 1$;
- $p = 3$: exponents 2 and 2, so $\min(2, 2) = 2$.

The primes 5, 7 appear in only one factorization and do not contribute.

Step 3: multiply.

$$\gcd(90, 126) = 2 \cdot 3^2 = 2 \cdot 9 = 18.$$

Check: $90 = 18 \cdot 5$ and $126 = 18 \cdot 7$, with $\gcd(5, 7) = 1$.

Example 5: $\gcd(150, 210)$

Step 1: factor both numbers by repeated division.

150	2	210	2
75	3	105	3
25	5	35	5
5	5	7	7
1		1	

Hence

$$150 = 2 \cdot 3 \cdot 5^2, \quad 210 = 2 \cdot 3 \cdot 5 \cdot 7.$$

Step 2: take the common primes with the smallest exponent.

- $p = 2$: exponents 1 and 1, so $\min(1, 1) = 1$;
- $p = 3$: exponents 1 and 1, so $\min(1, 1) = 1$;
- $p = 5$: exponents 2 and 1, so $\min(2, 1) = 1$.

The prime 7 appears in only one factorization and does not contribute.

Step 3: multiply.

$$\gcd(150, 210) = 2 \cdot 3 \cdot 5 = 30.$$

Check: $150 = 30 \cdot 5$ and $210 = 30 \cdot 7$, with $\gcd(5, 7) = 1$.

Example 6: $\gcd(180, 252)$

Step 1: factor both numbers by repeated division.

180	2	252	2
90	2	126	2
45	3	63	3
15	3	21	3
5	5	7	7
1		1	

Hence

$$180 = 2^2 \cdot 3^2 \cdot 5, \quad 252 = 2^2 \cdot 3^2 \cdot 7.$$

Step 2: take the common primes with the smallest exponent.

- $p = 2$: exponents 2 and 2, so $\min(2, 2) = 2$;
- $p = 3$: exponents 2 and 2, so $\min(2, 2) = 2$.

The primes 5, 7 appear in only one factorization and do not contribute.

Step 3: multiply.

$$\gcd(180, 252) = 2^2 \cdot 3^2 = 4 \cdot 9 = 36.$$

Check: $180 = 36 \cdot 5$ and $252 = 36 \cdot 7$, with $\gcd(5, 7) = 1$.

Example 7: $\gcd(231, 385)$

Step 1: factor both numbers by repeated division.

231	3	385	5
77	7	77	7
11	11	11	11
1		1	

Hence

$$231 = 3 \cdot 7 \cdot 11, \quad 385 = 5 \cdot 7 \cdot 11.$$

Step 2: take the common primes with the smallest exponent.

- $p = 7$: exponents 1 and 1, so $\min(1, 1) = 1$;
- $p = 11$: exponents 1 and 1, so $\min(1, 1) = 1$.

The primes 3, 5 appear in only one factorization and do not contribute.

Step 3: multiply.

$$\gcd(231, 385) = 7 \cdot 11 = 77.$$

Check: $231 = 77 \cdot 3$ and $385 = 77 \cdot 5$, with $\gcd(3, 5) = 1$.

Example 8: $\gcd(264, 396)$

Step 1: factor both numbers by repeated division.

264	2	396	2
132	2	198	2
66	2	99	3
33	3	33	3
11	11	11	11
1		1	

Hence

$$264 = 2^3 \cdot 3 \cdot 11, \quad 396 = 2^2 \cdot 3^2 \cdot 11.$$

Step 2: take the common primes with the smallest exponent.

- $p = 2$: exponents 3 and 2, so $\min(3, 2) = 2$;
- $p = 3$: exponents 1 and 2, so $\min(1, 2) = 1$;
- $p = 11$: exponents 1 and 1, so $\min(1, 1) = 1$.

Step 3: multiply.

$$\gcd(264, 396) = 2^2 \cdot 3 \cdot 11 = 4 \cdot 3 \cdot 11 = 132.$$

Check: $264 = 132 \cdot 2$ and $396 = 132 \cdot 3$, with $\gcd(2, 3) = 1$.

Example 9: $\gcd(315, 441)$

Step 1: factor both numbers by repeated division.

315	3	441	3
105	3	147	3
35	5	49	7
7	7	7	7
1		1	

Hence

$$315 = 3^2 \cdot 5 \cdot 7, \quad 441 = 3^2 \cdot 7^2.$$

Step 2: take the common primes with the smallest exponent.

- $p = 3$: exponents 2 and 2, so $\min(2, 2) = 2$;
- $p = 7$: exponents 1 and 2, so $\min(1, 2) = 1$.

The prime 5 appears in only one factorization and does not contribute.

Step 3: multiply.

$$\gcd(315, 441) = 3^2 \cdot 7 = 9 \cdot 7 = 63.$$

Check: $315 = 63 \cdot 5$ and $441 = 63 \cdot 7$, with $\gcd(5, 7) = 1$.

Example 10: $\gcd(504, 756)$

Step 1: factor both numbers by repeated division.

504	2	756	2
252	2	378	2
126	2	189	3
63	3	63	3
21	3	21	3
7	7	7	7
1		1	

Hence

$$504 = 2^3 \cdot 3^2 \cdot 7, \quad 756 = 2^2 \cdot 3^3 \cdot 7.$$

Step 2: take the common primes with the smallest exponent.

- $p = 2$: exponents 3 and 2, so $\min(3, 2) = 2$;
- $p = 3$: exponents 2 and 3, so $\min(2, 3) = 2$;
- $p = 7$: exponents 1 and 1, so $\min(1, 1) = 1$.

Step 3: multiply.

$$\gcd(504, 756) = 2^2 \cdot 3^2 \cdot 7 = 4 \cdot 9 \cdot 7 = 252.$$

Check: $504 = 252 \cdot 2$ and $756 = 252 \cdot 3$, with $\gcd(2, 3) = 1$.

2 Four-digit numbers

Example 11: $\gcd(1260, 3780)$

Step 1: factor both numbers by repeated division.

1260	2	3780	2
630	2	1890	2
315	3	945	3
105	3	315	3
35	5	105	3
7	7	35	5
1		7	7
		1	

Hence

$$1260 = 2^2 \cdot 3^2 \cdot 5 \cdot 7, \quad 3780 = 2^2 \cdot 3^3 \cdot 5 \cdot 7.$$

Step 2: take the common primes with the smallest exponent.

- $p = 2$: exponents 2 and 2, so $\min(2, 2) = 2$;
- $p = 3$: exponents 2 and 3, so $\min(2, 3) = 2$;
- $p = 5$: exponents 1 and 1, so $\min(1, 1) = 1$;
- $p = 7$: exponents 1 and 1, so $\min(1, 1) = 1$.

Step 3: multiply.

$$\gcd(1260, 3780) = 2^2 \cdot 3^2 \cdot 5 \cdot 7 = 4 \cdot 9 \cdot 5 \cdot 7 = 1260.$$

Check: $1260 = 1260 \cdot 1$ and $3780 = 1260 \cdot 3$, with $\gcd(1, 3) = 1$.

Example 12: $\gcd(2310, 3150)$

Step 1: factor both numbers by repeated division.

2310	2	3150	2
1155	3	1575	3
385	5	525	3
77	7	175	5
11	11	35	5
1		7	7
		1	

Hence

$$2310 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11, \quad 3150 = 2 \cdot 3^2 \cdot 5^2 \cdot 7.$$

Step 2: take the common primes with the smallest exponent.

- $p = 2$: exponents 1 and 1, so $\min(1, 1) = 1$;
- $p = 3$: exponents 1 and 2, so $\min(1, 2) = 1$;
- $p = 5$: exponents 1 and 2, so $\min(1, 2) = 1$;
- $p = 7$: exponents 1 and 1, so $\min(1, 1) = 1$.

The prime 11 appears in only one factorization and does not contribute.

Step 3: multiply.

$$\gcd(2310, 3150) = 2 \cdot 3 \cdot 5 \cdot 7 = 210.$$

Check: $2310 = 210 \cdot 11$ and $3150 = 210 \cdot 15$, with $\gcd(11, 15) = 1$.

Example 13: $\gcd(1848, 6552)$

Step 1: factor both numbers by repeated division.

1848	2	6552	2
924	2	3276	2
462	2	1638	2
231	3	819	3
77	7	273	3
11	11	91	7
1		13	13
		1	

Hence

$$1848 = 2^3 \cdot 3 \cdot 7 \cdot 11, \quad 6552 = 2^3 \cdot 3^2 \cdot 7 \cdot 13.$$

Step 2: take the common primes with the smallest exponent.

- $p = 2$: exponents 3 and 3, so $\min(3, 3) = 3$;
- $p = 3$: exponents 1 and 2, so $\min(1, 2) = 1$;
- $p = 7$: exponents 1 and 1, so $\min(1, 1) = 1$.

The primes 11, 13 appear in only one factorization and do not contribute.

Step 3: multiply.

$$\gcd(1848, 6552) = 2^3 \cdot 3 \cdot 7 = 8 \cdot 3 \cdot 7 = 168.$$

Check: $1848 = 168 \cdot 11$ and $6552 = 168 \cdot 39$, with $\gcd(11, 39) = 1$.

Example 14: $\gcd(4620, 7392)$

Step 1: factor both numbers by repeated division.

4620	2	7392	2
2310	2	3696	2
1155	3	1848	2
385	5	924	2
77	7	462	2
11	11	231	3
1		77	7
		11	11
		1	

Hence

$$4620 = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11, \quad 7392 = 2^5 \cdot 3 \cdot 7 \cdot 11.$$

Step 2: take the common primes with the smallest exponent.

- $p = 2$: exponents 2 and 5, so $\min(2, 5) = 2$;
- $p = 3$: exponents 1 and 1, so $\min(1, 1) = 1$;
- $p = 7$: exponents 1 and 1, so $\min(1, 1) = 1$;
- $p = 11$: exponents 1 and 1, so $\min(1, 1) = 1$.

The prime 5 appears in only one factorization and does not contribute.

Step 3: multiply.

$$\gcd(4620, 7392) = 2^2 \cdot 3 \cdot 7 \cdot 11 = 4 \cdot 3 \cdot 7 \cdot 11 = 924.$$

Check: $4620 = 924 \cdot 5$ and $7392 = 924 \cdot 8$, with $\gcd(5, 8) = 1$.

Example 15: $\gcd(3465, 5775)$

Step 1: factor both numbers by repeated division.

3465	3	5775	3
1155	3	1925	5
385	5	385	5
77	7	77	7
11	11	11	11
1		1	

Hence

$$3465 = 3^2 \cdot 5 \cdot 7 \cdot 11, \quad 5775 = 3 \cdot 5^2 \cdot 7 \cdot 11.$$

Step 2: take the common primes with the smallest exponent.

- $p = 3$: exponents 2 and 1, so $\min(2, 1) = 1$;
- $p = 5$: exponents 1 and 2, so $\min(1, 2) = 1$;
- $p = 7$: exponents 1 and 1, so $\min(1, 1) = 1$;
- $p = 11$: exponents 1 and 1, so $\min(1, 1) = 1$.

Step 3: multiply.

$$\gcd(3465, 5775) = 3 \cdot 5 \cdot 7 \cdot 11 = 1155.$$

Check: $3465 = 1155 \cdot 3$ and $5775 = 1155 \cdot 5$, with $\gcd(3, 5) = 1$.
