

FCS Math0 - Second Lesson

Computing the GCD by the Euclidean Algorithm

Caboara

Method

The algorithm rests on the following fact: if $a = qb + r$ with $0 \leq r < b$, then

$$\gcd(a, b) = \gcd(b, r),$$

because every common divisor of a and b divides $r = a - qb$, and every common divisor of b and r divides $a = qb + r$.

To compute $\gcd(a, b)$ with $a \geq b > 0$:

1. Divide a by b : $a = q_1b + r_1$, with $0 \leq r_1 < b$.
2. If $r_1 \neq 0$, divide b by r_1 , then r_1 by r_2 , and so on, each time dividing the previous divisor by the previous remainder.
3. The remainders strictly decrease, so eventually one is 0. The *last nonzero remainder* is $\gcd(a, b)$ (and $\gcd(n, 0) = n$).

1 Two- and three-digit numbers

Example 1: $\gcd(84, 120)$

Since $120 > 84$, we start by dividing 120 by 84.

Successive divisions with remainder.

$120 = 1 \cdot 84 + 36$	remainder $36 \neq 0$
$84 = 2 \cdot 36 + 12$	remainder $12 \neq 0$
$36 = 3 \cdot 12 + 0$	remainder 0: stop

The last nonzero remainder is 12, obtained after 3 divisions. Since $\gcd(x, y) = \gcd(y, r)$ at each step,

$$\gcd(120, 84) = \gcd(84, 36) = \gcd(36, 12) = \gcd(12, 0) = 12.$$

Check: $84 = 12 \cdot 7$ and $120 = 12 \cdot 10$, with $\gcd(7, 10) = 1$.

Example 2: $\gcd(36, 60)$

Since $60 > 36$, we start by dividing 60 by 36.

Successive divisions with remainder.

$60 = 1 \cdot 36 + 24$	remainder $24 \neq 0$
$36 = 1 \cdot 24 + 12$	remainder $12 \neq 0$
$24 = 2 \cdot 12 + 0$	remainder 0: stop

The last nonzero remainder is 12, obtained after 3 divisions. Since $\gcd(x, y) = \gcd(y, r)$ at each step,

$$\gcd(60, 36) = \gcd(36, 24) = \gcd(24, 12) = \gcd(12, 0) = 12.$$

Check: $36 = 12 \cdot 3$ and $60 = 12 \cdot 5$, with $\gcd(3, 5) = 1$.

Example 3: $\gcd(72, 96)$

Since $96 > 72$, we start by dividing 96 by 72.

Successive divisions with remainder.

$96 = 1 \cdot 72 + 24$	remainder $24 \neq 0$
$72 = 3 \cdot 24 + 0$	remainder 0: stop

The last nonzero remainder is 24, obtained after 2 divisions. Since $\gcd(x, y) = \gcd(y, r)$ at each step,

$$\gcd(96, 72) = \gcd(72, 24) = \gcd(24, 0) = 24.$$

Check: $72 = 24 \cdot 3$ and $96 = 24 \cdot 4$, with $\gcd(3, 4) = 1$.

Example 4: $\gcd(90, 126)$

Since $126 > 90$, we start by dividing 126 by 90.

Successive divisions with remainder.

$126 = 1 \cdot 90 + 36$	remainder $36 \neq 0$
$90 = 2 \cdot 36 + 18$	remainder $18 \neq 0$
$36 = 2 \cdot 18 + 0$	remainder 0: stop

The last nonzero remainder is 18, obtained after 3 divisions. Since $\gcd(x, y) = \gcd(y, r)$ at each step,

$$\gcd(126, 90) = \gcd(90, 36) = \gcd(36, 18) = \gcd(18, 0) = 18.$$

Check: $90 = 18 \cdot 5$ and $126 = 18 \cdot 7$, with $\gcd(5, 7) = 1$.

Example 5: $\gcd(150, 210)$

Since $210 > 150$, we start by dividing 210 by 150.

Successive divisions with remainder.

$210 = 1 \cdot 150 + 60$	remainder $60 \neq 0$
$150 = 2 \cdot 60 + 30$	remainder $30 \neq 0$
$60 = 2 \cdot 30 + 0$	remainder 0: stop

The last nonzero remainder is 30, obtained after 3 divisions. Since $\gcd(x, y) = \gcd(y, r)$ at each step,

$$\gcd(210, 150) = \gcd(150, 60) = \gcd(60, 30) = \gcd(30, 0) = 30.$$

Check: $150 = 30 \cdot 5$ and $210 = 30 \cdot 7$, with $\gcd(5, 7) = 1$.

Example 6: $\gcd(180, 252)$

Since $252 > 180$, we start by dividing 252 by 180.

Successive divisions with remainder.

$$\begin{array}{ll} 252 = 1 \cdot 180 + 72 & \text{remainder } 72 \neq 0 \\ 180 = 2 \cdot 72 + 36 & \text{remainder } 36 \neq 0 \\ 72 = 2 \cdot 36 + 0 & \text{remainder } 0: \text{ stop} \end{array}$$

The last nonzero remainder is 36, obtained after 3 divisions. Since $\gcd(x, y) = \gcd(y, r)$ at each step,

$$\gcd(252, 180) = \gcd(180, 72) = \gcd(72, 36) = \gcd(36, 0) = 36.$$

Check: $180 = 36 \cdot 5$ and $252 = 36 \cdot 7$, with $\gcd(5, 7) = 1$.

Example 7: $\gcd(231, 385)$

Since $385 > 231$, we start by dividing 385 by 231.

Successive divisions with remainder.

$$\begin{array}{ll} 385 = 1 \cdot 231 + 154 & \text{remainder } 154 \neq 0 \\ 231 = 1 \cdot 154 + 77 & \text{remainder } 77 \neq 0 \\ 154 = 2 \cdot 77 + 0 & \text{remainder } 0: \text{ stop} \end{array}$$

The last nonzero remainder is 77, obtained after 3 divisions. Since $\gcd(x, y) = \gcd(y, r)$ at each step,

$$\gcd(385, 231) = \gcd(231, 154) = \gcd(154, 77) = \gcd(77, 0) = 77.$$

Check: $231 = 77 \cdot 3$ and $385 = 77 \cdot 5$, with $\gcd(3, 5) = 1$.

Example 8: $\gcd(264, 396)$

Since $396 > 264$, we start by dividing 396 by 264.

Successive divisions with remainder.

$$\begin{array}{ll} 396 = 1 \cdot 264 + 132 & \text{remainder } 132 \neq 0 \\ 264 = 2 \cdot 132 + 0 & \text{remainder } 0: \text{ stop} \end{array}$$

The last nonzero remainder is 132, obtained after 2 divisions. Since $\gcd(x, y) = \gcd(y, r)$ at each step,

$$\gcd(396, 264) = \gcd(264, 132) = \gcd(132, 0) = 132.$$

Check: $264 = 132 \cdot 2$ and $396 = 132 \cdot 3$, with $\gcd(2, 3) = 1$.

Example 9: $\gcd(315, 441)$

Since $441 > 315$, we start by dividing 441 by 315.

Successive divisions with remainder.

$$\begin{array}{ll} 441 = 1 \cdot 315 + 126 & \text{remainder } 126 \neq 0 \\ 315 = 2 \cdot 126 + 63 & \text{remainder } 63 \neq 0 \\ 126 = 2 \cdot 63 + 0 & \text{remainder } 0: \text{ stop} \end{array}$$

The last nonzero remainder is 63, obtained after 3 divisions. Since $\gcd(x, y) = \gcd(y, r)$ at each step,

$$\gcd(441, 315) = \gcd(315, 126) = \gcd(126, 63) = \gcd(63, 0) = 63.$$

Check: $315 = 63 \cdot 5$ and $441 = 63 \cdot 7$, with $\gcd(5, 7) = 1$.

Example 10: $\gcd(504, 756)$

Since $756 > 504$, we start by dividing 756 by 504.

Successive divisions with remainder.

$$\begin{array}{ll} 756 = 1 \cdot 504 + 252 & \text{remainder } 252 \neq 0 \\ 504 = 2 \cdot 252 + 0 & \text{remainder } 0: \text{ stop} \end{array}$$

The last nonzero remainder is 252, obtained after 2 divisions. Since $\gcd(x, y) = \gcd(y, r)$ at each step,

$$\gcd(756, 504) = \gcd(504, 252) = \gcd(252, 0) = 252.$$

Check: $504 = 252 \cdot 2$ and $756 = 252 \cdot 3$, with $\gcd(2, 3) = 1$.

2 Four-digit numbers

Example 11: $\gcd(1260, 3780)$

Since $3780 > 1260$, we start by dividing 3780 by 1260.

Successive divisions with remainder.

$$3780 = 3 \cdot 1260 + 0 \quad \text{remainder } 0: \text{ stop}$$

The remainder is 0 at the first step, so 1260 divides 3780 and

$$\gcd(3780, 1260) = \gcd(1260, 0) = 1260.$$

Check: $1260 = 1260 \cdot 1$ and $3780 = 1260 \cdot 3$, with $\gcd(1, 3) = 1$.

Example 12: $\gcd(2310, 3150)$

Since $3150 > 2310$, we start by dividing 3150 by 2310.

Successive divisions with remainder.

$$\begin{array}{ll} 3150 = 1 \cdot 2310 + 840 & \text{remainder } 840 \neq 0 \\ 2310 = 2 \cdot 840 + 630 & \text{remainder } 630 \neq 0 \\ 840 = 1 \cdot 630 + 210 & \text{remainder } 210 \neq 0 \\ 630 = 3 \cdot 210 + 0 & \text{remainder } 0: \text{ stop} \end{array}$$

The last nonzero remainder is 210, obtained after 4 divisions. Since $\gcd(x, y) = \gcd(y, r)$ at each step,

$$\gcd(3150, 2310) = \gcd(2310, 840) = \gcd(840, 630) = \gcd(630, 210) = \gcd(210, 0) = 210.$$

Check: $2310 = 210 \cdot 11$ and $3150 = 210 \cdot 15$, with $\gcd(11, 15) = 1$.

Example 13: $\gcd(1848, 6552)$

Since $6552 > 1848$, we start by dividing 6552 by 1848.

Successive divisions with remainder.

$6552 = 3 \cdot 1848 + 1008$	remainder $1008 \neq 0$
$1848 = 1 \cdot 1008 + 840$	remainder $840 \neq 0$
$1008 = 1 \cdot 840 + 168$	remainder $168 \neq 0$
$840 = 5 \cdot 168 + 0$	remainder 0: stop

The last nonzero remainder is 168, obtained after 4 divisions. Since $\gcd(x, y) = \gcd(y, r)$ at each step,

$$\gcd(6552, 1848) = \gcd(1848, 1008) = \gcd(1008, 840) = \gcd(840, 168) = \gcd(168, 0) = 168.$$

Check: $1848 = 168 \cdot 11$ and $6552 = 168 \cdot 39$, with $\gcd(11, 39) = 1$.

Example 14: $\gcd(4620, 7392)$

Since $7392 > 4620$, we start by dividing 7392 by 4620.

Successive divisions with remainder.

$7392 = 1 \cdot 4620 + 2772$	remainder $2772 \neq 0$
$4620 = 1 \cdot 2772 + 1848$	remainder $1848 \neq 0$
$2772 = 1 \cdot 1848 + 924$	remainder $924 \neq 0$
$1848 = 2 \cdot 924 + 0$	remainder 0: stop

The last nonzero remainder is 924, obtained after 4 divisions. Since $\gcd(x, y) = \gcd(y, r)$ at each step,

$$\gcd(7392, 4620) = \gcd(4620, 2772) = \gcd(2772, 1848) = \gcd(1848, 924) = \gcd(924, 0) = 924.$$

Check: $4620 = 924 \cdot 5$ and $7392 = 924 \cdot 8$, with $\gcd(5, 8) = 1$.

Example 15: $\gcd(3465, 5775)$

Since $5775 > 3465$, we start by dividing 5775 by 3465.

Successive divisions with remainder.

$5775 = 1 \cdot 3465 + 2310$	remainder $2310 \neq 0$
$3465 = 1 \cdot 2310 + 1155$	remainder $1155 \neq 0$
$2310 = 2 \cdot 1155 + 0$	remainder 0: stop

The last nonzero remainder is 1155, obtained after 3 divisions. Since $\gcd(x, y) = \gcd(y, r)$ at each step,

$$\gcd(5775, 3465) = \gcd(3465, 2310) = \gcd(2310, 1155) = \gcd(1155, 0) = 1155.$$

Check: $3465 = 1155 \cdot 3$ and $5775 = 1155 \cdot 5$, with $\gcd(3, 5) = 1$.
