

FCS MATH0 Second Lesson

Classwork Polynomial Long Division

Caboara

The method

Given two polynomials $A(x)$ (the *dividend*) and $B(x) \neq 0$ (the *divisor*), there exist unique polynomials $Q(x)$ (the *quotient*) and $R(x)$ (the *remainder*) such that

$$A(x) = B(x)Q(x) + R(x), \quad R(x) = 0 \text{ or } \deg R < \deg B.$$

To find them we proceed as follows.

1. Write $A(x)$ and $B(x)$ in decreasing powers of x , inserting the missing powers in $A(x)$ with coefficient 0 (this keeps the columns aligned). The dividend goes on the left, the divisor on the right, above a horizontal line; the quotient is built below that line.
2. Divide the leading term of the current dividend by the leading term of $B(x)$: the result is the next term of the quotient.
3. Multiply the whole divisor $B(x)$ by this term, write the product under the current dividend *with all signs changed*, and add column by column. The leading term cancels (marked //), so the degree drops.
4. The sum is the new current dividend. Repeat steps 2–3 until its degree is smaller than $\deg B$: what is left is the remainder $R(x)$.

Finally, it is good practice to check the result by verifying that $B(x)Q(x) + R(x) = A(x)$.

Exercise 1: $(x^3 - x + 1) : (x - 3)$

We write the dividend with all powers: $x^3 + 0x^2 - x + 1$.

- **Step 1.** $x^3 : x = x^2$. $x^2(x - 3) = x^3 - 3x^2$; with changed signs: $-x^3 + 3x^2$.
Add: $(x^3 + 0x^2 - x + 1) + (-x^3 + 3x^2) = 3x^2 - x + 1$.
- **Step 2.** $3x^2 : x = 3x$. $3x(x - 3) = 3x^2 - 9x$; with changed signs: $-3x^2 + 9x$.
Add: $(3x^2 - x + 1) + (-3x^2 + 9x) = 8x + 1$.
- **Step 3.** $8x : x = 8$. $8(x - 3) = 8x - 24$; with changed signs: $-8x + 24$.
Add: $(8x + 1) + (-8x + 24) = 25$.

The remainder 25 has degree $0 < 1 = \deg(x - 3)$, so we stop.

$$\begin{array}{r|l}
x^3 & +0x^2 & -x & +1 & x-3 \\
-x^3 & +3x^2 & & & \hline
\hline
// & 3x^2 & -x & +1 & \\
& -3x^2 & +9x & & \\
\hline
& // & 8x & +1 & \\
& & -8x & +24 & \\
\hline
& & // & 25 &
\end{array}$$

$$Q(x) = x^2 + 3x + 8, \quad R(x) = 25.$$

Check. $(x-3)(x^2+3x+8) + 25 = x^3 + 3x^2 + 8x - 3x^2 - 9x - 24 + 25 = x^3 - x + 1. \checkmark$

Remark. Since the divisor is $x-3$, by the Remainder Theorem $R = A(3) = 27 - 3 + 1 = 25$, which confirms the result.

Exercise 2: $(x^3 + x^2 + x - 1) : (x^2 + x)$

All powers are already present in the dividend; the divisor is $x^2 + x + 0$.

- **Step 1.** $x^3 : x^2 = x$. $x(x^2 + x) = x^3 + x^2$; with changed signs: $-x^3 - x^2$.
Add: $(x^3 + x^2 + x - 1) + (-x^3 - x^2) = x - 1$.

The remainder $x - 1$ has degree $1 < 2 = \deg(x^2 + x)$, so we stop after a single step.

$$\begin{array}{r|l}
x^3 & +x^2 & +x & -1 & x^2+x \\
-x^3 & -x^2 & & & \hline
\hline
// & // & x & -1 &
\end{array}$$

$$Q(x) = x, \quad R(x) = x - 1.$$

Check. $(x^2 + x) \cdot x + (x - 1) = x^3 + x^2 + x - 1. \checkmark$

Exercise 3: $(x^4 - x^3 + 2) : (x^2 + x + 1)$

We write the dividend with all powers: $x^4 - x^3 + 0x^2 + 0x + 2$.

- **Step 1.** $x^4 : x^2 = x^2$. $x^2(x^2 + x + 1) = x^4 + x^3 + x^2$; with changed signs: $-x^4 - x^3 - x^2$.
Add: $(x^4 - x^3 + 0x^2 + 0x + 2) + (-x^4 - x^3 - x^2) = -2x^3 - x^2 + 0x + 2$.
- **Step 2.** $-2x^3 : x^2 = -2x$. $-2x(x^2 + x + 1) = -2x^3 - 2x^2 - 2x$; with changed signs: $2x^3 + 2x^2 + 2x$.
Add: $(-2x^3 - x^2 + 0x + 2) + (2x^3 + 2x^2 + 2x) = x^2 + 2x + 2$.
- **Step 3.** $x^2 : x^2 = 1$. $1 \cdot (x^2 + x + 1) = x^2 + x + 1$; with changed signs: $-x^2 - x - 1$.
Add: $(x^2 + 2x + 2) + (-x^2 - x - 1) = x + 1$.

The remainder $x + 1$ has degree $1 < 2 = \deg(x^2 + x + 1)$, so we stop.

$$\begin{array}{r|l}
\begin{array}{rrrrr}
x^4 & -x^3 & +0x^2 & +0x & +2 \\
-x^4 & -x^3 & -x^2 & & \\
\hline
// & -2x^3 & -x^2 & +0x & +2 \\
& +2x^3 & +2x^2 & +2x & \\
\hline
& // & x^2 & +2x & +2 \\
& & -x^2 & -x & -1 \\
& & \hline
& & // & x & +1
\end{array}
&
\begin{array}{l}
x^2 + x + 1 \\
\hline
x^2 - 2x + 1
\end{array}
\end{array}$$

$$Q(x) = x^2 - 2x + 1 = (x - 1)^2, \quad R(x) = x + 1.$$

Check.

$$\begin{aligned}
(x^2 + x + 1)(x^2 - 2x + 1) + (x + 1) &= x^4 - 2x^3 + x^2 + x^3 - 2x^2 + x + x^2 - 2x + 1 + x + 1 \\
&= x^4 - x^3 + 0x^2 + 0x + 2 = x^4 - x^3 + 2. \quad \checkmark
\end{aligned}$$