Trigonometric equations

A summary of the solving rules

Caboara

Conventions and tools

- Angles are measured in radians. The letter k always denotes an arbitrary integer, $k \in \mathbb{Z}$: writing $x = \frac{\pi}{6} + 2k\pi$ means that the solutions are all the numbers $\frac{\pi}{6} + 2k\pi$, for every $k \in \mathbb{Z}$.
- We do not use the inverse trigonometric functions: all the equations lead to the *notable values* of the table below. We do not use the prosthaphaeresis (sum-to-product) formulas.
- Before solving, write the *domain* of the equation: $\tan x$ requires $x \neq \frac{\pi}{2} + k\pi$, a denominator must be nonzero. At the end, discard the solutions outside the domain.

Notable values

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan x$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	not defined

Associated angles and periodicity

$$\begin{array}{lll}
\sin(-x) = -\sin x & \cos(-x) = \cos x & \tan(-x) = -\tan x \\
\sin(\pi - x) = \sin x & \cos(\pi - x) = -\cos x & \tan(\pi - x) = -\tan x \\
\sin(\pi + x) = -\sin x & \cos(\pi + x) = -\cos x & \tan(\pi + x) = \tan x \\
\sin\left(\frac{\pi}{2} - x\right) = \cos x & \cos\left(\frac{\pi}{2} - x\right) = \sin x & \tan\left(\frac{\pi}{2} - x\right) = \dfrac{1}{\tan x} \\
\sin\left(\frac{\pi}{2} + x\right) = \cos x & \cos\left(\frac{\pi}{2} + x\right) = -\sin x & \tan\left(\frac{\pi}{2} + x\right) = -\dfrac{1}{\tan x}
\end{array}$$

The functions sin and cos have period 2π, the function tan has period π. The two formulas for $\tan\left(\frac{\pi}{2} \pm x\right)$ hold for $x \neq \frac{k\pi}{2}$. For instance $\sin\frac{5\pi}{6} = \sin\left(\pi - \frac{\pi}{6}\right) = \frac{1}{2}$ and $\cos\frac{2\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = -\frac{1}{2}$.

Identities

- Fundamental identities: $\sin^2 x + \cos^2 x = 1$, $\quad \tan x = \dfrac{\sin x}{\cos x}$, $\quad 1 + \tan^2 x = \dfrac{1}{\cos^2 x}$.
- Addition formulas:

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta, \qquad \cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

- Duplication formulas:

$$\sin 2x = 2\sin x\cos x, \qquad \cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

- Parametric formulas (tangent of the half angle). For $x \neq \pi + 2k\pi$, setting $t = \tan\frac{x}{2}$,

$$\sin x = \frac{2t}{1+t^2}, \qquad \cos x = \frac{1-t^2}{1+t^2}$$

Indeed $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \dfrac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}$ and $\cos x = \dfrac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}$; dividing numerator and denominator by $\cos^2\frac{x}{2}$, which is nonzero because $\frac{x}{2} \neq \frac{\pi}{2} + k\pi$, we obtain the formulas.

1 Elementary equations

Rule 1 $(\sin x = a)$**.**

- If $a > 1$ or $a < -1$ the equation has no solutions.
- If $-1 \leq a \leq 1$, let α be the angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with $\sin\alpha = a$. Then

$$\sin x = a \iff x = \alpha + 2k\pi \quad \text{or} \quad x = \pi - \alpha + 2k\pi$$

On the unit circle: the two points with ordinate a, symmetric with respect to the y-axis.

- Special cases, with a single family of solutions: $\sin x = 0 \iff x = k\pi$, $\sin x = 1 \iff x = \frac{\pi}{2} + 2k\pi$, $\sin x = -1 \iff x = -\frac{\pi}{2} + 2k\pi$.
- If the argument is an expression $f(x)$, apply the rule to $f(x)$, and then solve for x: $\sin f(x) = a \iff f(x) = \alpha + 2k\pi$ or $f(x) = \pi - \alpha + 2k\pi$.

Example 1. Solve $2\sin\left(2x - \frac{\pi}{3}\right) = \sqrt{3}$.

Solution. We have $\sin\left(2x - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} = \sin\frac{\pi}{3}$, so $\alpha = \frac{\pi}{3}$ and

$$2x - \frac{\pi}{3} = \frac{\pi}{3} + 2k\pi \quad \text{or} \quad 2x - \frac{\pi}{3} = \pi - \frac{\pi}{3} + 2k\pi = \frac{2\pi}{3} + 2k\pi$$

From the first equation $2x = \frac{2\pi}{3} + 2k\pi$, that is $x = \frac{\pi}{3} + k\pi$; from the second $2x = \pi + 2k\pi$, that is $x = \frac{\pi}{2} + k\pi$. Notice that dividing by 2 the period $2k\pi$ becomes $k\pi$.
Solutions: $x = \frac{\pi}{3} + k\pi$ or $x = \frac{\pi}{2} + k\pi$. □

Rule 2 $(\cos x = a)$**.**

- If $a > 1$ or $a < -1$ the equation has no solutions.
- If $-1 \leq a \leq 1$, let α be the angle in $[0, \pi]$ with $\cos\alpha = a$. Then

$$\cos x = a \iff x = \alpha + 2k\pi \quad \text{or} \quad x = -\alpha + 2k\pi \qquad (\text{in short } x = \pm\alpha + 2k\pi)$$

On the unit circle: the two points with abscissa a, symmetric with respect to the x-axis.

- Special cases: $\cos x = 0 \iff x = \frac{\pi}{2} + k\pi$, $\cos x = 1 \iff x = 2k\pi$, $\cos x = -1 \iff x = \pi + 2k\pi$.

- As before, $\cos f(x) = a \iff f(x) = \pm\alpha + 2k\pi$.

Example 2. Solve $2\cos\frac{x}{2} + \sqrt{2} = 0$.

Solution. We have $\cos\frac{x}{2} = -\frac{\sqrt{2}}{2}$. Since $\cos\left(\pi - \frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$, the angle in $[0, \pi]$ is $\alpha = \frac{3\pi}{4}$, and

$$\frac{x}{2} = \pm\frac{3\pi}{4} + 2k\pi \iff x = \pm\frac{3\pi}{2} + 4k\pi$$

Multiplying by 2 the period $2k\pi$ becomes $4k\pi$.
Solutions: $x = \frac{3\pi}{2} + 4k\pi$ or $x = -\frac{3\pi}{2} + 4k\pi$. □

Rule 3 ($\tan x = a$)**.**

- Domain of $\tan x$: $x \neq \frac{\pi}{2} + k\pi$. For every real number a, let α be the angle in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with $\tan\alpha = a$. Then

  $$\tan x = a \iff x = \alpha + k\pi$$

  A single family of solutions, with period π.

- As before, $\tan f(x) = a \iff f(x) = \alpha + k\pi$, with the domain condition $f(x) \neq \frac{\pi}{2} + k\pi$.

Example 3. Solve $\sqrt{3}\tan 3x - 1 = 0$.

Solution. Domain: $3x \neq \frac{\pi}{2} + k\pi$. We have $\tan 3x = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} = \tan\frac{\pi}{6}$, so

$$3x = \frac{\pi}{6} + k\pi \iff x = \frac{\pi}{18} + \frac{k\pi}{3}$$

For these values $3x = \frac{\pi}{6} + k\pi$, which is never of the form $\frac{\pi}{2} + h\pi$: the domain condition is satisfied.
Solutions: $x = \frac{\pi}{18} + \frac{k\pi}{3}$. □

2 Equations reducible to elementary ones

Rule 4 (Same function on both sides)**.** If A and B are expressions in x,

$$\begin{aligned}
\sin A = \sin B &\iff A = B + 2k\pi \quad \text{or} \quad A = \pi - B + 2k\pi \\
\cos A = \cos B &\iff A = B + 2k\pi \quad \text{or} \quad A = -B + 2k\pi \\
\tan A = \tan B &\iff A = B + k\pi \qquad \text{with } A \neq \tfrac{\pi}{2} + h\pi,\ B \neq \tfrac{\pi}{2} + h\pi
\end{aligned}$$

Then solve the resulting equations for x. If x disappears from one of them, that equation is either impossible or always true. For instance in $\sin\left(x + \frac{\pi}{3}\right) = \sin x$ the first equation is $x + \frac{\pi}{3} = x + 2k\pi$, that is $\frac{\pi}{3} = 2k\pi$, which is impossible: only the second one gives solutions.

Example 4. Solve $\sin 3x = \sin\left(x + \frac{\pi}{4}\right)$.

Solution.

- $3x = x + \frac{\pi}{4} + 2k\pi \iff 2x = \frac{\pi}{4} + 2k\pi \iff x = \frac{\pi}{8} + k\pi$.
- $3x = \pi - \left(x + \frac{\pi}{4}\right) + 2k\pi \iff 4x = \frac{3\pi}{4} + 2k\pi \iff x = \frac{3\pi}{16} + \frac{k\pi}{2}$.

Solutions: $x = \frac{\pi}{8} + k\pi$ or $x = \frac{3\pi}{16} + \frac{k\pi}{2}$. □

Rule 5 (Different functions or opposite signs)**.** Use the associated angles to obtain the same function on both sides, and then apply Rule 4:

$$\sin A = \cos B \iff \sin A = \sin\left(\tfrac{\pi}{2} - B\right) \qquad \cos A = \sin B \iff \cos A = \cos\left(\tfrac{\pi}{2} - B\right)$$
$$\sin A = -\sin B \iff \sin A = \sin(-B) \qquad \cos A = -\cos B \iff \cos A = \cos(\pi - B)$$
$$\sin A = -\cos B \iff \sin A = \sin\left(B - \tfrac{\pi}{2}\right) \qquad \tan A = -\tan B \iff \tan A = \tan(-B)$$
$$\tan A = \frac{1}{\tan B} \iff \tan A = \tan\left(\tfrac{\pi}{2} - B\right)$$

(with the domain conditions when tan appears).

Example 5. Solve $\cos 2x = \sin x$.

Solution. Since $\sin x = \cos\left(\frac{\pi}{2} - x\right)$ the equation becomes $\cos 2x = \cos\left(\frac{\pi}{2} - x\right)$, and

- $2x = \frac{\pi}{2} - x + 2k\pi \iff 3x = \frac{\pi}{2} + 2k\pi \iff x = \frac{\pi}{6} + \frac{2k\pi}{3}$;
- $2x = -\frac{\pi}{2} + x + 2k\pi \iff x = -\frac{\pi}{2} + 2k\pi$.

The second family is contained in the first one: $-\frac{\pi}{2} + 2k\pi = \frac{\pi}{6} + \frac{2(3k-1)\pi}{3}$. In $[0, 2\pi)$ the solutions are $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$.
Solutions: $x = \frac{\pi}{6} + \frac{2k\pi}{3}$. □

Remark. The same equation can be solved with Rule 7: $\cos 2x = 1 - 2\sin^2 x$, so $2\sin^2 x + \sin x - 1 = 0$, that is $\sin x = \frac{1}{2}$ or $\sin x = -1$, which gives again $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$ (plus $2k\pi$). Different methods can give the solutions written in different, but equivalent, ways.

3 Algebraic methods

Rule 6 (Algebraic equations in one trigonometric function)**.** If the equation contains only $\sin x$ (or only $\cos x$, or only $\tan x$), set $t = \sin x$ (or $t = \cos x$, or $t = \tan x$) and

1. solve the algebraic equation in t;
2. keep only the values with $-1 \le t \le 1$ when $t = \sin x$ or $t = \cos x$ (every real value is allowed when $t = \tan x$);
3. solve the elementary equations $\sin x = t_1$, $\sin x = t_2$, … with Rules 1–3.

Example 6. Solve $2\sin^2 x - 5\sin x + 2 = 0$.

Solution. With $t = \sin x$ we get $2t^2 - 5t + 2 = 0$, so $t = \frac{5 \pm \sqrt{25-16}}{4} = \frac{5 \pm 3}{4}$, that is $t = 2$ or $t = \frac{1}{2}$. The value $t = 2$ is discarded, since $\sin x \le 1$. From $\sin x = \frac{1}{2} = \sin\frac{\pi}{6}$ we get $x = \frac{\pi}{6} + 2k\pi$ or $x = \pi - \frac{\pi}{6} + 2k\pi = \frac{5\pi}{6} + 2k\pi$.
Solutions: $x = \frac{\pi}{6} + 2k\pi$ or $x = \frac{5\pi}{6} + 2k\pi$. □

Rule 7 (Reduction to one function and one argument)**.** If different functions or different arguments appear, use the identities to obtain an equation in a single function of a single argument, and then apply Rule 6. Typical substitutions:

- $\cos^2 x = 1 - \sin^2 x$ and $\sin^2 x = 1 - \cos^2 x$;
- $\cos 2x = 1 - 2\sin^2 x$ (if the other terms contain $\sin x$) or $\cos 2x = 2\cos^2 x - 1$ (if they contain $\cos x$);
- $\sin(x \pm \alpha)$, $\cos(x \pm \alpha)$ expanded with the addition formulas;

- $\frac{1}{\cos^2 x} = 1 + \tan^2 x$, with the domain conditions.

Example 7. Solve $\cos 2x + 3\cos x + 2 = 0$.

Solution. The other terms contain $\cos x$, so we use $\cos 2x = 2\cos^2 x - 1$:

$$2\cos^2 x - 1 + 3\cos x + 2 = 0 \iff 2\cos^2 x + 3\cos x + 1 = 0$$

With $t = \cos x$, $2t^2 + 3t + 1 = 0$ gives $t = \frac{-3\pm 1}{4}$, that is $t = -\frac{1}{2}$ or $t = -1$. Then $\cos x = -\frac{1}{2} = \cos\frac{2\pi}{3}$ gives $x = \pm\frac{2\pi}{3} + 2k\pi$, and $\cos x = -1$ gives $x = \pi + 2k\pi$.
Solutions: $x = \pm\frac{2\pi}{3} + 2k\pi$ or $x = \pi + 2k\pi$. □

Rule 8 (Factorization)**.** Bring all the terms to the same side and factor:

$$A \cdot B = 0 \iff A = 0 \quad \text{or} \quad B = 0$$

(with both factors defined). Never divide both sides by an expression containing x that may vanish: you would lose the solutions where it vanishes.

Example 8. Solve $\sin 2x = \sqrt{3}\sin x$.

Solution. Using $\sin 2x = 2\sin x\cos x$,

$$2\sin x\cos x - \sqrt{3}\sin x = 0 \iff \sin x\left(2\cos x - \sqrt{3}\right) = 0$$

From $\sin x = 0$ we get $x = k\pi$; from $\cos x = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6}$ we get $x = \pm\frac{\pi}{6} + 2k\pi$. Dividing both sides by $\sin x$ we would have lost the solutions $x = k\pi$.
Solutions: $x = k\pi$ or $x = \pm\frac{\pi}{6} + 2k\pi$. □

4 Equations in $\sin x$ and $\cos x$

Rule 9 (Homogeneous equations)**.**

(a) First degree: $a\sin x + b\cos x = 0$, with $a, b \neq 0$. The values with $\cos x = 0$ are not solutions (for them $\sin x = \pm 1$, and $a\sin x \neq 0$), so we can divide by $\cos x$:

$$a\tan x + b = 0 \iff \tan x = -\frac{b}{a}$$

(b) Second degree: $a\sin^2 x + b\sin x\cos x + c\cos^2 x = 0$.

- If $a \neq 0$, the values with $\cos x = 0$ are not solutions (for them $\sin^2 x = 1$ and the left side is $a \neq 0$): divide by $\cos^2 x$ and solve $a\tan^2 x + b\tan x + c = 0$ with Rule 6.
- If $a = 0$, factor: $\cos x\,(b\sin x + c\cos x) = 0$ (Rule 8). Here $\cos x = 0$ gives solutions $x = \frac{\pi}{2} + k\pi$.

(c) Reducible to homogeneous: $a\sin^2 x + b\sin x\cos x + c\cos^2 x = d$. Write the constant as $d = d\left(\sin^2 x + \cos^2 x\right)$ and bring everything to the left:

$$(a-d)\sin^2 x + b\sin x\cos x + (c-d)\cos^2 x = 0$$

Example 9. Solve $\sin^2 x + 2\sqrt{3}\sin x\cos x - \cos^2 x = 2$.

Solution. Writing $2 = 2\sin^2 x + 2\cos^2 x$ and bringing everything to the left,

$$-\sin^2 x + 2\sqrt{3}\sin x\cos x - 3\cos^2 x = 0 \iff \sin^2 x - 2\sqrt{3}\sin x\cos x + 3\cos^2 x = 0$$

The coefficient of $\sin^2 x$ is nonzero, so $\cos x = 0$ gives no solutions, and we divide by $\cos^2 x$:

$$\tan^2 x - 2\sqrt{3}\tan x + 3 = 0 \iff \left(\tan x - \sqrt{3}\right)^2 = 0 \iff \tan x = \sqrt{3}$$

that is $x = \frac{\pi}{3} + k\pi$.
Solutions: $x = \frac{\pi}{3} + k\pi$. □

Remark. A first degree example: $\sqrt{3}\sin x - \cos x = 0 \iff \tan x = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \iff x = \frac{\pi}{6} + k\pi$.

Rule 10 (Linear equations: tangent of the half angle)**.** Consider $a\sin x + b\cos x = c$ with $a, b, c \neq 0$ (if $c = 0$ the equation is homogeneous, Rule 9; if $a = 0$ or $b = 0$ it is elementary).

1. *Check* $x = \pi + 2k\pi$, where $\tan\frac{x}{2}$ is not defined. Since $\sin\pi = 0$ and $\cos\pi = -1$, these values are solutions if and only if $-b = c$.

2. *For* $x \neq \pi + 2k\pi$ set $t = \tan\frac{x}{2}$ and use the parametric formulas:

   $$a\,\frac{2t}{1+t^2} + b\,\frac{1-t^2}{1+t^2} = c$$

   Multiplying by $1 + t^2 > 0$ we get

   $$(b+c)\,t^2 - 2a\,t + (c-b) = 0$$

3. *Solve for* t, and then $\tan\frac{x}{2} = t = \tan\alpha \iff \frac{x}{2} = \alpha + k\pi \iff x = 2\alpha + 2k\pi$.

4. The solutions are those found in step 1 together with those found in step 3.

If $b + c = 0$ the equation in t has degree one: this is exactly the case in which $x = \pi + 2k\pi$ is a solution, which the substitution loses. That is why step 1 is necessary. If the equation in t has negative discriminant (and $b + c \neq 0$) there are no solutions.

Example 10. Solve $\sqrt{3}\sin x - \cos x = 1$.

Solution. Here $a = \sqrt{3}$, $b = -1$, $c = 1$.

1. For $x = \pi$: $\sqrt{3}\cdot 0 - (-1) = 1$. So $x = \pi + 2k\pi$ are solutions.

2. For $x \neq \pi + 2k\pi$, with $t = \tan\frac{x}{2}$:

   $$\sqrt{3}\,\frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2} = 1 \iff 2\sqrt{3}\,t - 1 + t^2 = 1 + t^2 \iff 2\sqrt{3}\,t = 2 \iff t = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

   The term in t^2 cancels, as expected since $b + c = 0$.

3. $\tan\frac{x}{2} = \frac{\sqrt{3}}{3} = \tan\frac{\pi}{6} \iff \frac{x}{2} = \frac{\pi}{6} + k\pi \iff x = \frac{\pi}{3} + 2k\pi$.

Solutions: $x = \frac{\pi}{3} + 2k\pi$ or $x = \pi + 2k\pi$. □

Remark. We can check the result with the addition formula, without inverse functions:

$$\sqrt{3}\sin x - \cos x = 2\left(\sin x\cdot\frac{\sqrt{3}}{2} - \cos x\cdot\frac{1}{2}\right) = 2\left(\sin x\cos\frac{\pi}{6} - \cos x\sin\frac{\pi}{6}\right) = 2\sin\left(x - \frac{\pi}{6}\right)$$

The equation becomes $\sin\left(x - \frac{\pi}{6}\right) = \frac{1}{2}$, so $x - \frac{\pi}{6} = \frac{\pi}{6} + 2k\pi$ or $x - \frac{\pi}{6} = \frac{5\pi}{6} + 2k\pi$, that is $x = \frac{\pi}{3} + 2k\pi$ or $x = \pi + 2k\pi$, as before.

How to choose the rule

1. One function of one argument, alone: Rules 1, 2, 3.
2. The same function on both sides, with different arguments: Rule 4; different functions or a minus sign: Rule 5.
3. One function, but in a polynomial expression: Rule 6.
4. Different functions or arguments that can be reduced to one with the identities: Rule 7.
5. A common factor, or an expression that can be factored: Rule 8.
6. All terms of the same degree in $\sin x$ and $\cos x$ (possibly after writing a constant d as $d(\sin^2 x + \cos^2 x)$): Rule 9.
7. $a \sin x + b \cos x = c$ with $c \neq 0$: Rule 10.

Always remember the domain, and never divide by an expression that may vanish.