The Graph of $A\,F(Bx + C) + D$

Caboara

Let $F\colon \operatorname{dom} F \subseteq \mathbb{R} \to \mathbb{R}$ be a function and let $A, B, C, D \in \mathbb{R}$ with $A \neq 0$, $B \neq 0$. We study the graph of

$$G(x) = A\,F(Bx + C) + D$$

in terms of the graph of F. (If $A = 0$, G is the constant D; if $B = 0$, G is the constant $A\,F(C) + D$.)

1 The general rule

Proposition 1. *A point (u, v) lies on the graph of F if and only if the point*

$$\left(\frac{u - C}{B},\ A\,v + D\right)$$

lies on the graph of G. Hence the graph of G is obtained from the graph of F by moving every point (u, v) to $\left(\frac{u-C}{B},\ Av + D\right)$.

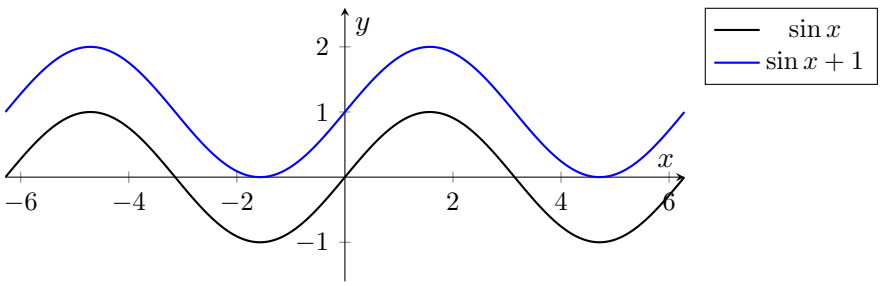
Proof. Put $x = \frac{u-C}{B}$, that is $u = Bx + C$. Then (x, y) is on the graph of G if and only if $y = A\,F(Bx + C) + D = A\,F(u) + D$, if and only if $\frac{y-D}{A} = F(u)$, if and only if $\left(u, \frac{y-D}{A}\right)$ is on the graph of F. Writing $v = \frac{y-D}{A}$, that is $y = Av + D$, we get the statement. □

In words: the parameters *outside* F (A, D) act on the y coordinate *as written*; the parameters *inside* F (B, C) act on the x coordinate *in the opposite way* (subtract C, divide by B).

2 The parameters one at a time

D: vertical translation, $y = F(x) + D$

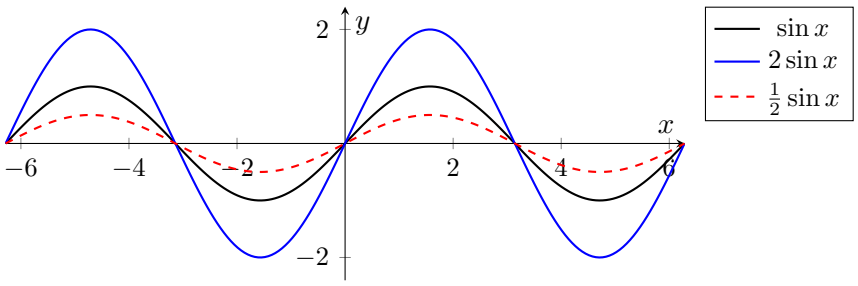
The graph is translated *up* by D if $D > 0$, *down* by $|D|$ if $D < 0$. Each point (u, v) goes to $(u, v + D)$.



A: vertical stretch and reflection, $y = A\,F(x)$

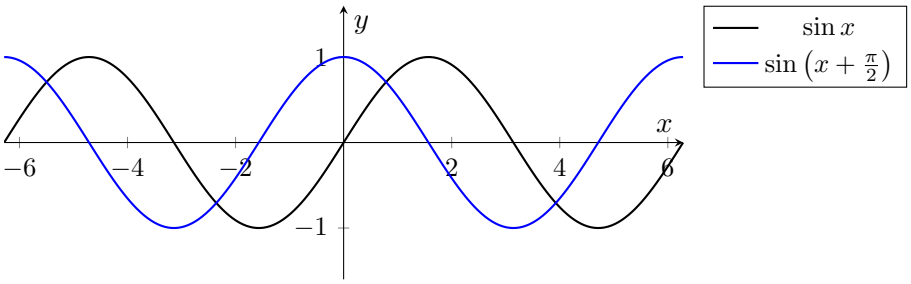
Each point (u, v) goes to (u, Av); the points on the x-axis (the zeros of F) do not move.

- $|A| > 1$: vertical *stretch* by the factor $|A|$;
- $0 < |A| < 1$: vertical *compression* by the factor $|A|$;
- $A < 0$: in addition, *reflection* in the x-axis. In particular $-F(x)$ is the reflection of $F(x)$ in the x-axis.



C: horizontal translation, $y = F(x + C)$

Each point (u, v) goes to $(u - C, v)$: the graph is translated to the *left* by C if $C > 0$, to the *right* by $|C|$ if $C < 0$. This is the opposite of what one might expect: $F(x + C)$ takes at x the value that F takes at $x + C$, that is, C units further to the right.



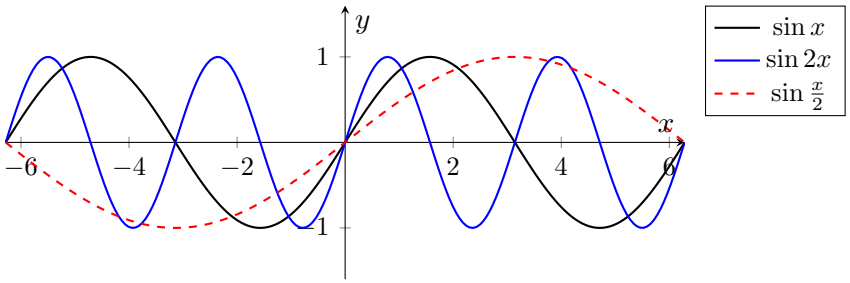
Here the graph moves $\frac{\pi}{2}$ to the left, and indeed $\sin\left(x + \frac{\pi}{2}\right) = \cos x$.

B: horizontal stretch and reflection, $y = F(Bx)$

Each point (u, v) goes to $\left(\frac{u}{B}, v\right)$; the points on the y-axis do not move.

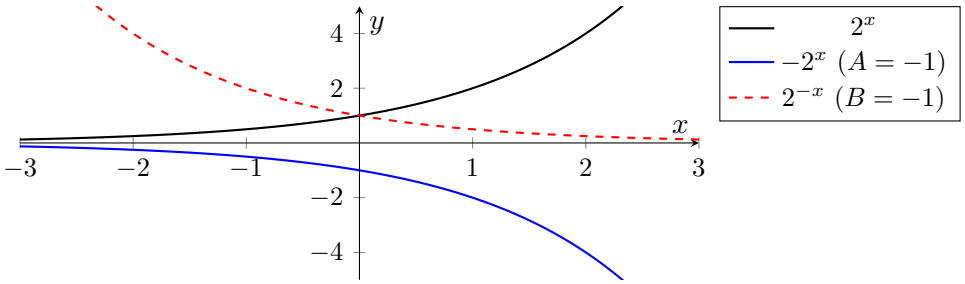
- $|B| > 1$: horizontal *compression* by the factor $|B|$ (the graph is squeezed towards the y-axis);
- $0 < |B| < 1$: horizontal *stretch* by the factor $\frac{1}{|B|}$;
- $B < 0$: in addition, *reflection* in the y-axis. In particular $F(-x)$ is the reflection of $F(x)$ in the y-axis.

Again the effect is the opposite of what the formula suggests: multiplying x by 2 makes the graph *narrower*.



The two reflections

Since sin is odd, its reflections in the two axes coincide. To see the difference we use $F(x) = 2^x$.



3 All the parameters together

To draw $G(x) = A\,F(Bx + C) + D$ starting from the graph of F, perform in this order:

1. $F(x + C)$: translate horizontally by $-C$;
2. $F(Bx + C)$: compress horizontally by the factor $|B|$ (reflecting in the y-axis if $B < 0$);
3. $A\,F(Bx + C)$: stretch vertically by the factor $|A|$ (reflecting in the x-axis if $A < 0$);
4. $A\,F(Bx + C) + D$: translate vertically by D.

Remark (The horizontal shift is $-\frac{C}{B}$, not $-C$)**.** *Writing* $Bx + C = B\left(x + \frac{C}{B}\right)$*, the graph of* G *is also obtained by first compressing by* $|B|$ *and* then *translating horizontally by* $-\frac{C}{B}$*. For instance* $\sin(2x - \frac{\pi}{3}) = \sin\left(2\left(x - \frac{\pi}{6}\right)\right)$ *is the graph of* $\sin 2x$ *moved* $\frac{\pi}{6}$ *to the right, not* $\frac{\pi}{3}$*. The vertical operations, on the other hand, are done as written: first multiply by* A*, then add* D*.*

What happens to the main features

From the general rule:

- **Domain:** $x \in \operatorname{dom} G \iff Bx + C \in \operatorname{dom} F$.
- **Image:** $\operatorname{Im} G = \{A\,y + D \mid y \in \operatorname{Im} F\}$. For example, if $\operatorname{Im} F = [m, M]$ and $A > 0$, then $\operatorname{Im} G = [Am + D, AM + D]$.
- **Zeros:** the zeros of F move to $\frac{u-C}{B}$ only when $D = 0$; if $D \neq 0$ the zeros must be recomputed.
- **Period:** if F is periodic of period T, then G is periodic of period $\frac{T}{|B|}$.
- **Monotonicity:** on corresponding intervals, G is monotone in the same way as F if $AB > 0$, in the opposite way if $AB < 0$.
- **Maxima and minima:** a maximum point (u, v) of F goes to $\left(\frac{u-C}{B}, Av + D\right)$, which is a maximum of G if $A > 0$ and a minimum if $A < 0$.

The sinusoids

For $F = \sin$ (or cos) the parameters have names:

$$y = A\sin(Bx + C) + D: \quad \begin{array}{ll} |A| & \text{amplitude,} \\ \frac{2\pi}{|B|} & \text{period,} \\ -\frac{C}{B} & \text{phase shift (horizontal translation),} \\ D & \text{midline } y = D, \text{ image } [D - |A|,\ D + |A|]. \end{array}$$

Example. $y = 3\sin\left(2x - \frac{\pi}{3}\right) + 1$ *has amplitude* 3*, period* π*, phase shift* $\frac{\pi}{6}$ *to the right, midline* $y = 1$ *and image* $[-2, 4]$*. Its maxima are at the points where* $2x - \frac{\pi}{3} = \frac{\pi}{2} + 2k\pi$*, that is* $x = \frac{5\pi}{12} + k\pi$*, with value* 4*.*

