Even and odd functions

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1 Symmetric domains

To compare $f(x)$ and $f(-x)$ we need both to exist. A set $D \subseteq \mathbb{R}$ is *symmetric with respect to* 0 if

$$x \in D \implies -x \in D$$

For instance $\mathbb{R}$, $[-2, 2]$, $\mathbb{R} \setminus \{0\}$ are symmetric; $[0, +\infty)$, $[-1, 3]$, $\mathbb{R} \setminus \{1\}$ are not.

2 Definitions

Definition 1. Let f be a function whose domain D is symmetric with respect to 0.

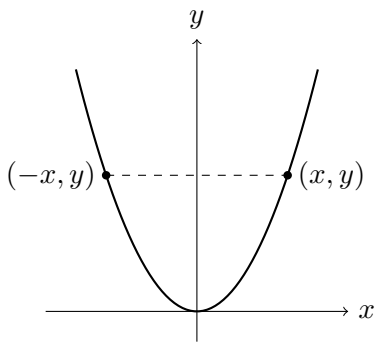
- f is *even* if $f(-x) = f(x)$ for every $x \in D$;
- f is *odd* if $f(-x) = -f(x)$ for every $x \in D$.

If the domain is not symmetric, f is neither even nor odd.

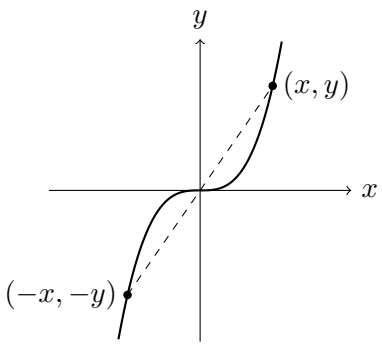
Remark. Most functions are neither even nor odd. To show that f is *not* even it is enough to find one value x_0 with $f(-x_0) \neq f(x_0)$; to show that it is *not* odd, one value with $f(-x_0) \neq -f(x_0)$.

3 Graphs

- The graph of an even function is symmetric with respect to the y-axis: if (x, y) is on the graph, so is $(-x, y)$.
- The graph of an odd function is symmetric with respect to the origin: if (x, y) is on the graph, so is $(-x, -y)$.



$y = x^2$, even



$y = x^3$, odd

4 Examples

Example 1 (Powers)**.** $f(x) = x^n$, with n a positive integer, has domain $\mathbb{R}$ and $f(-x) = (-x)^n = (-1)^n x^n$.

- If n is even, $(-1)^n = 1$ and f is even: x^2, x^4, ...
- If n is odd, $(-1)^n = -1$ and f is odd: x, x^3, ...

This is the origin of the names. A constant function $f(x) = c$ is even.

Example 2 (Polynomials)**.** $f(x) = 3x^4 - x^2 + 5$ contains only even powers (the constant $5 = 5x^0$ counts as an even power): $f(-x) = 3x^4 - x^2 + 5 = f(x)$, so f is even. $g(x) = 2x^5 - 7x^3 + x$ contains only odd powers: $g(-x) = -2x^5 + 7x^3 - x = -g(x)$, so g is odd.

$h(x) = x^2 + x$ contains both: $h(1) = 2$ and $h(-1) = 0$. Since $h(-1) \neq h(1)$, h is not even; since $h(-1) \neq -h(1)$, h is not odd.

Example 3 (Absolute value)**.** $f(x) = |x|$: $f(-x) = |-x| = |x| = f(x)$, so f is even.

Example 4 (Trigonometric functions)**.** $\cos(-x) = \cos x$, so cos is even; $\sin(-x) = -\sin x$, so sin is odd. The domain of tan, $x \neq \frac{\pi}{2} + k\pi$, is symmetric, and

$$\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin x}{\cos x} = -\tan x$$

so tan is odd.

Example 5 (Fractions)**.** $f(x) = \frac{1}{x}$ has the symmetric domain $\mathbb{R}\setminus\{0\}$ and $f(-x) = -\frac{1}{x} = -f(x)$: f is odd. The function $g(x) = \frac{1}{x-1}$ has domain $\mathbb{R} \setminus \{1\}$, which is not symmetric (-1 belongs to it, 1 does not): g is neither even nor odd.

Example 6 (Domain first)**.** $f(x) = \sqrt{x}$ has domain $[0, +\infty)$, which is not symmetric: f is neither even nor odd. Instead $g(x) = \sqrt{9 - x^2}$ has domain $[-3, 3]$ and $g(-x) = \sqrt{9 - (-x)^2} = g(x)$: g is even.

5 Properties

Property 1 (Products and quotients)**.** For functions with the same symmetric domain (and nonzero denominators):

$$\text{even} \cdot \text{even} = \text{even}, \qquad \text{odd} \cdot \text{odd} = \text{even}, \qquad \text{even} \cdot \text{odd} = \text{odd}$$

and the same for quotients (the rule is the same as for the signs + and −).

Proof. For instance, if f and g are odd and $p(x) = f(x)g(x)$, then $p(-x) = f(-x)g(-x) = (-f(x))(-g(x)) = f(x)g(x) = p(x)$. The other cases are similar. □

For instance $x \sin x$ is even (odd times odd), $x^2 \sin x$ is odd (even times odd), $\frac{\cos x}{x}$ is odd.

Property 2 (Sums)**.** The sum of two even functions is even, the sum of two odd functions is odd. The sum of an even and an odd function, both nonzero, is in general neither: for instance $x^2 + x$ (Example 2).

Property 3 (Composition)**.** If g is even then $f(g(x))$ is even: $f(g(-x)) = f(g(x))$. For instance $\cos(x^2)$, $|x|^3$ and $(\sin x)^2 = \sin^2 x$ are even; $\sin^2 x$ is the composition of $t \mapsto t^2$ with sin, and $(\sin(-x))^2 = (-\sin x)^2 = \sin^2 x$.

Property 4 (Value at 0)**.** If f is odd and 0 is in its domain, then $f(0) = 0$: from $f(-0) = -f(0)$ we get $f(0) = -f(0)$, that is $2f(0) = 0$. For instance $f(x) = x^3 + 1$ is not odd, since $f(0) = 1$.

Property 5 (Even and odd at the same time)**.** The only function, on a given symmetric domain, which is both even and odd is the zero function: from $f(-x) = f(x)$ and $f(-x) = -f(x)$ we get $f(x) = -f(x)$, that is $f(x) = 0$.

How to check the parity of a function

1. Find the domain. If it is not symmetric, the function is neither even nor odd.
2. Compute $f(-x)$ and simplify.
3. If you obtain $f(x)$, the function is even; if you obtain $-f(x)$, it is odd.
4. Otherwise, find a value x_0 which shows that it is neither: $f(-x_0) \neq f(x_0)$ and $f(-x_0) \neq -f(x_0)$.