Even and odd functions

Exercises with solutions

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The exercises go from easy to medium. Remember: first the domain, then $f(-x)$.

Exercise 1. Say whether each function is even, odd or neither.

$$f(x) = x^4 - 3x^2 + 2, \qquad g(x) = x^5 - 2x, \qquad h(x) = x^2 + x$$

Solution. All three functions have domain $\mathbb{R}$, which is symmetric.

- $f(-x) = (-x)^4 - 3(-x)^2 + 2 = x^4 - 3x^2 + 2 = f(x)$: f is **even**.
- $g(-x) = (-x)^5 - 2(-x) = -x^5 + 2x = -(x^5 - 2x) = -g(x)$: g is **odd**.
- $h(-x) = x^2 - x$, which is neither $h(x)$ nor $-h(x)$. To be sure, take $x_0 = 1$: $h(1) = 2$, $h(-1) = 0$. Since $0 \neq 2$, h is not even; since $0 \neq -2$, h is not odd. So h is **neither**.

□

Exercise 2. Say whether each function is even, odd or neither.

$$f(x) = |x| - x^2, \qquad g(x) = \frac{x^3}{x^2 + 1}, \qquad h(x) = x^3 + 1$$

Solution. The three functions have domain $\mathbb{R}$ (for g, $x^2 + 1 > 0$ for every x).

- $f(-x) = |-x| - (-x)^2 = |x| - x^2 = f(x)$: f is **even**.
- $g(-x) = \dfrac{(-x)^3}{(-x)^2 + 1} = \dfrac{-x^3}{x^2 + 1} = -g(x)$: g is **odd**. (Odd divided by even.)
- $h(0) = 1 \neq 0$, so h is not odd (an odd function defined at 0 has $f(0) = 0$). Moreover $h(1) = 2$ and $h(-1) = 0$, so h is not even. So h is **neither**.

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Exercise 3. Say whether each function is even, odd or neither.

$$f(x) = x \sin x, \qquad g(x) = x^2 \sin x, \qquad h(x) = \sin x + \cos x$$

Solution. The domain is $\mathbb{R}$. Recall that $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$.

- $f(-x) = (-x)\sin(-x) = (-x)(-\sin x) = x \sin x = f(x)$: f is **even**.
- $g(-x) = (-x)^2 \sin(-x) = x^2(-\sin x) = -g(x)$: g is **odd**.
- Take $x_0 = \frac{\pi}{2}$: $h\left(\frac{\pi}{2}\right) = 1 + 0 = 1$ and $h\left(-\frac{\pi}{2}\right) = -1 + 0 = -1$. Since $-1 \neq 1$, h is not even. Take $x_0 = 0$: $h(0) = 1 \neq 0$, so h is not odd. So h is **neither**.

□

Exercise 4. Find the domain and say whether each function is even, odd or neither.

$$f(x) = \sqrt{4 - x^2}, \qquad g(x) = \frac{1}{x - 1}, \qquad h(x) = \frac{x}{x^2 - 4}$$

Solution.

- f: we need $4 - x^2 \geq 0$, that is $-2 \leq x \leq 2$. The domain $[-2, 2]$ is symmetric, and $f(-x) = \sqrt{4 - (-x)^2} = \sqrt{4 - x^2} = f(x)$: f is **even**.
- g: the domain is $\mathbb{R} \setminus \{1\}$. It is not symmetric: -1 is in the domain, 1 is not. So g is **neither** even nor odd.
- h: we need $x^2 \neq 4$, so the domain is $\mathbb{R} \setminus \{-2, 2\}$, which is symmetric. $h(-x) = \dfrac{-x}{(-x)^2 - 4} = \dfrac{-x}{x^2 - 4} = -h(x)$: h is **odd**.

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Exercise 5. The function $f\colon \mathbb{R} \to \mathbb{R}$ is odd, and $f(x) = x^2 - 2x$ for $x \geq 0$.

(a) Compute $f(3)$ and $f(-3)$.

(b) Write the expression of $f(x)$ for $x < 0$.

Solution.

(a) $f(3) = 9 - 6 = 3$. Since f is odd, $f(-3) = -f(3) = -3$.

(b) Let $x < 0$. Then $-x > 0$, so we can use the given formula for $f(-x)$:

$$f(-x) = (-x)^2 - 2(-x) = x^2 + 2x$$

Since f is odd, $f(x) = -f(-x) = -x^2 - 2x$. So

$$f(x) = \begin{cases} x^2 - 2x & x \geq 0 \\ -x^2 - 2x & x < 0 \end{cases}$$

Check: for $x = -3$ the second formula gives $-9 + 6 = -3$, as in (a). Notice also $f(0) = 0$, as it must be for an odd function.

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Exercise 6.

(a) Prove that the product of two odd functions (with the same symmetric domain) is even.

(b) Using (a) and the other rules for products, say whether the following functions are even, odd or neither:

$$p(x) = \sin x \tan x, \qquad q(x) = x^3 \cos x, \qquad r(x) = \frac{\sin x}{x}$$

Solution.

(a) Let f and g be odd and $p(x) = f(x)g(x)$. Then

$$p(-x) = f(-x)\, g(-x) = \big(-f(x)\big)\big(-g(x)\big) = f(x)g(x) = p(x)$$

so p is even.

(b)

- $p(x) = \sin x \tan x$: domain $x \neq \frac{\pi}{2} + k\pi$, symmetric. sin and tan are odd, so by (a) p is **even**. Directly: $p(-x) = (-\sin x)(-\tan x) = \sin x \tan x$.
- $q(x) = x^3 \cos x$: domain $\mathbb{R}$. x^3 is odd and $\cos x$ is even, so q is **odd**: $q(-x) = (-x^3)\cos x = -q(x)$.
- $r(x) = \frac{\sin x}{x}$: domain $\mathbb{R} \setminus \{0\}$, symmetric. Odd divided by odd, so r is **even**: $r(-x) = \frac{-\sin x}{-x} = \frac{\sin x}{x} = r(x)$.

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