Homework

Caboara

Exercise 1. *Prove the trigonometric identity* $\cos x \cos 2x \cos 4x = \dfrac{1}{8}$.

Solution. **Step 1: the case** $\sin x = 0$**.** If $\sin x = 0$, then $x = m\pi$, so $\cos x = \pm 1$, $\cos 2x = 1$, $\cos 4x = 1$, and the left-hand side equals $\pm 1 \neq \frac{1}{8}$. Hence every solution satisfies $\sin x \neq 0$.

Step 2: multiply by $8 \sin x$**.** For $\sin x \neq 0$ the equation is equivalent to

$$8 \sin x \cos x \cos 2x \cos 4x = \sin x.$$

Apply the duplication formula $2 \sin \alpha \cos \alpha = \sin 2\alpha$ three times:

$$8 \sin x \cos x \cos 2x \cos 4x = 4 \sin 2x \cos 2x \cos 4x = 2 \sin 4x \cos 4x = \sin 8x.$$

So the equation becomes

$$\sin 8x = \sin x, \qquad \sin x \neq 0.$$

Step 3: solve $\sin 8x = \sin x$**.** Two angles have the same sine if and only if they are equal, or supplementary, up to multiples of 2π:

$$8x = x + 2k\pi \qquad \text{or} \qquad 8x = \pi - x + 2k\pi,$$

that is

$$x = \frac{2k\pi}{7} \qquad \text{or} \qquad x = \frac{(2k+1)\pi}{9}.$$

Step 4: discard the values with $\sin x = 0$**.**

- $x = \frac{2k\pi}{7}$ is a multiple of π if and only if k is a multiple of 7, i.e. the remainder of k in the division by 7 is 0: these values must be excluded.
- $x = \frac{(2k+1)\pi}{9}$ is a multiple of π if and only if $2k+1$ is a multiple of 9. Since $2k+1$ is odd, this means $2k + 1 = 9(2m + 1)$, i.e. $k = 9m + 4$: these are the integers k whose remainder in the division by 9 is 4, and they must be excluded.

Conclusion. The solutions are

$$x = \frac{2k\pi}{7} \qquad \text{and} \qquad x = \frac{(2k+1)\pi}{9},$$

where in the first family k is any integer whose remainder in the division by 7 is different from 0, and in the second family k is any integer whose remainder in the division by 9 is different from 4.

A consistency check. In $[0, 2\pi)$ the first family gives 6 solutions ($k = 1, \ldots, 6$) and the second gives 8 solutions ($k = 0, \ldots, 8$, $k \neq 4$): 14 in total. This agrees with the fact that $\cos x \cos 2x \cos 4x$ is a polynomial of degree 7 in $\cos x$; hence $\cos x$ can take at most 7 values, each attained exactly twice in $[0, 2\pi)$ when it differs from ± 1, which we already excluded. □