NUMERICAL SOLUTION OF A MATRIX INTEGRAL EQUATION ARISING IN MARKOV MODULATED LÉVY PROCESSES*

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Abstract. Markov-modulated Lévy processes lead to matrix integral equations of the kind $A_0 + A_1 X + A_2 X^2 + A_3(X) = 0$ where A_0, A_1, A_2 are given matrix coefficients, while $A_3(X)$ is a nonlinear function, expressed in terms of integrals involving the exponential of the matrix X itself. In this paper we propose some numerical methods for the solution of this class of matrix equations, perform a theoretical convergence analysis and show the effectiveness of the new methods by means of a wide numerical experimentation.

Key words. Markov modulation, Lévy processes, Nonlinear matrix equations, Regime switching, Integral equations, Numerical algorithms, Fixed-point iterations, Algebraic Riccati equation

MSC: 65F45, 60J22, 15A24, 60G51, 60J65.

1. Introduction.

1.1. The problem. A Markov-modulated Lévy process $\{(X_t, J_t)\}$ has two components: X_t takes real values and is called the level, J_t is called the phase, it is a continuous-time Markov process with irreducible generator Q and state space $\{1, \ldots, n\}$. Over intervals of time where $J(t)$ remains constant, and equal to i say, the level behaves as the superposition of a Brownian motion and Poisson processes of jumps, with parameters dependent on i; when the phases switches from i to $j \neq i$, $X(t)$ jumps by a random quantity Y_{ij}. Applications are found in mathematical finance (Jobert and Rogers [18], Ballotta and Bonfiglioli [1], Deelstra and Simon [11]), risk theory (Lu and Tsai [22], Li and Ren [21]) and queueing theory (Prabhu [25]).

Such a process is characterised by the matrix-valued function $F(s)$ such that $F_{ij}(s) = \mathrm{E}[e^{sX_t}\mathbb{I}[J_t = j]|X_0 = 0, J_0 = i]$, for $t \geq 0$, $1 \leq i,j \leq n$ and s such that the expectation is finite, where $\mathbb{I}[\cdot]$ is the indicator function; $F(s)$ is given by

$$(1.1) \qquad F(s) = \mathrm{diag}(\psi_1(s), \ldots, \psi_n(s)) + \int_{\mathbb{R}} e^{sx}(Q \circ U(dx))$$

where $\psi_i(s)$ is the Laplace exponent of the Lévy process corresponding to phase i, $U_{ij}(\cdot)$, $i \neq j$, is a probability distribution function on $\mathbb{R}$ and $U_{ii}(\cdot)$ is the step function: $U_{ii}(x) = 0$ for $x < 0$, $U_{ii}(x) = 1$ for $x \geq 0$ (see d'Auria *et al.* [10]). We define $A \circ B$ as the Hadamard product of A and B.

Jumps during an interval of sojourn in a given phase are parametrised by the so-called Lévy measure (Bertoin [3, Page 3]) which we assume to be finite for all phases. In physical terms, this means that jumps occur at epochs of Poisson processes, instead of a denumerably infinite superposition of Poisson processes. This assumption is not very restrictive when one considers practical applications, and it will make it easier to implement numerical algorithms.

Furthermore, we consider that jumps take positive values only. In that case, a matrix denoted as G plays a fundamental role in the analysis of the Markov-modulated Lévy process, it is such that $(e^{Gx})_{ij}$ is the probability that the phase is j upon the first visit to level $-x$, given that the process starts from level 0, in phase i at time 0.

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Few authors have considered the question of computing G. Breuer [7, Theorem 2] suggests a functional iteration procedure, but there is no indication that the author has implemented it, and he does not perform a convergence analysis. D'Auria *et al.* [10] conduct a spectral analysis of the matrix function $F(s)$ and obtain a characterisation of the Jordan normal form of G. This, in principle, may serve as the basis for a numerical procedure, but they do not perform it. Moreover, we point out that a numerical procedure based on the computation of the Jordan normal form of a matrix is sensitive to numerical error and is deprecated for that reason. Simon [30, Proposition 4.8] defines a different functional iteration procedure, there is no implementation, nor convergence analysis. One should add that these authors consider processes with negative jumps only, but our results are easily adapted to that case.

The $n \times n$ matrix G solves an integral matrix equation of the kind

$$(1.2) \qquad\qquad A_0 + A_1 G + A_2 G^2 + A_3(G) = 0$$

where A_0, A_1, A_2 are given $n \times n$ matrix coefficients, while $A_3(G)$ is a nonlinear matrix function, expressed in terms of integrals involving the exponential of the matrix G.

1.2. The results. Our purpose in the present paper is to define and analyse numerical procedures to compute G by solving the matrix equation (1.2). To our knowledge, a numerical analysis of matrix equations of the kind (1.2) has not been performed in the literature, while there is wide literature on matrix equations of the kind $\sum_{i=0}^{\infty} A_i G^i = 0$. We refer to the books [5] and [20] for a survey on matrix equations arising in structured Markov chains, to [16] and [17] for the solution of general quadratic matrix equations, to [23] for an analysis of the conditioning, and to [6], [8], [9], [13], [14], [24], [26], [28], and [29] for strategies to improve the accuracy and the convergence.

The numerical methods that we propose, inspired by the recent paper [6], consist in solving a sequence of quadratic matrix equations, where the matrix coefficients defining the matrix equation depend on the current approximation to the solution G. The numerical methods differ in the way the quadratic matrix equations are generated. For instance, a natural approach consists in generating the sequence $\{G_k\}_k$, where G_{k+1} solves the quadratic matrix equation $A_0 + A_1 X + A_2 X^2 + A_3(G_k) = 0$, for $k = 0, 1, \ldots$. We elaborate this idea and propose different strategies for defining the equation and for solving it. A preliminary step consists in applying a change of variable, so that the solution of the new equation is stochastic. In the first method we obtain a suitable unilateral quadratic matrix equation, depending on a positive parameter τ, with nonnegativity properties of the matrix coefficients. In the second method, we obtain a special nonsymmetric algebraic Riccati equation. A theoretical analysis of the conditions of convergence and of the convergence rate is carried out, together with the determination of the optimal value of τ for the first method. Numerical tests are performed together with comparisons with the algorithms proposed by Breuer [7] and Simon [30]. The numerical results confirm the theoretical analysis and show the effectiveness of our approach with respect to [7] and [30].

The paper is organized as follows. In Section 2 we recall some properties on nonnegative matrices and introduce some notation. In Section 3 we describe the matrix integral equation and recall some properties of the solution G. In Sections 4 and 5 we introduce the changes of variable and we rewrite the matrix integral equation in terms of a unilateral quadratic matrix equation and of a nonsymmetric algebraic Riccati equation, respectively, the matrix coefficients of which depend on G itself. Section 6 is devoted to the design of three numerical methods for computing G,

based on the different reductions to quadratic and Riccati matrix equations. Finally, numerical results are presented and discussed in Section 7.

2. Notation and preliminaries. A real matrix $A = (a_{i,j})_{i,j}$ is called nonnegative (positive), and we write $A \geq 0$ $(A > 0)$, if $a_{i,j} \geq 0$ $(a_{i,j} > 0)$ for every i, j. We write $A \geq B$ $(A > B)$ if $A - B \geq 0$ $(A - B > 0)$. The spectral radius of a matrix A is denoted by $\rho(A)$. A real matrix A is called an M-matrix if it can be written in the form $A = \sigma I - B$ where $B \geq 0$ and $\rho(B) \leq \sigma$. A real matrix A is called a Z-matrix if $a_{ij} \leq 0$ for $i \neq j$. The following result gives sufficient conditions under which a Z-matrix is an M-matrix (see [2, Chapter 6, Lemma 4.1, Theorem 2.3 and Exercise 5.1c]):

THEOREM 2.1. *Assume that A is a Z-matrix. If there exists a vector $u > 0$ such that $Au \geq 0$ then A is an M-matrix; if in addition $Au > 0$, or if A is irreducible and $Au \neq 0$, then A is a nonsingular M-matrix. If A is an M-matrix and $B \geq A$ is a Z-matrix, then B is an M-matrix. Finally, if A is a nonsingular M-matrix then $A^{-1} \geq 0$.* $\square$

Given a real matrix A, the splitting $A = M - N$ is said to be a regular splitting if $\det M \neq 0$, $M^{-1} \geq 0$ and $N \geq 0$. The following result provides spectral properties of $M^{-1}N$ (see [32, Theorems 3.13-3.15])

THEOREM 2.2. *Let A be a real matrix such that $A^{-1} \geq 0$. If $A = M - N$ is a regular splitting then $\rho(M^{-1}N) < 1$. If $A = M_1 - N_1$ and $A = M_2 - N_2$ are two regular splittings such that $N_1 \leq N_2$, then $\rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2)$; if, in addition, $A^{-1} > 0$ and $N_1 \neq N_2$ then $\rho(M_1^{-1}N_1) < \rho(M_2^{-1}N_2)$.* $\square$

A generator $A = (a_{i,j})_{i,j}$ is a square real matrix such that $a_{i,j} \geq 0$ for $i \neq j$ and $A\mathbf{1} = 0$, where $\mathbf{1}$ is the vector with all entries equal to 1. In particular, $-A$ is a singular M-matrix. The matrix A is a subgenerator if $a_{i,j} \geq 0$ for $i \neq j$ and $A\mathbf{1} \leq 0$.

For any square matrices A and E, and for any $t \geq 0$, the following identity holds [15, Section 10.2]

$$(2.1) \qquad e^{(A+E)t} - e^{At} = \int_0^t e^{A(t-s)} E e^{(A+E)s} ds.$$

Given a vector $y = (y_i)_{i=1,\ldots,n}$, we denote by Δ_y the $n \times n$ diagonal matrix with diagonal entries $y_1, \ldots, y_n$. Given two matrices $A = (a_{i,j})_{ij}$ and $B = (b_{i,j})_{ij}$ having the same dimension, the expression $C = A \circ B$ denotes the Hadamard product, where the entries of C are $c_{i,j} = a_{i,j}b_{i,j}$.

3. The matrix integral equation. In this paper, the expression (1.1) for the matrix function $F(s)$ is more precisely the following:

$$F(s) = \frac{1}{2}\Delta_{\sigma^2}s^2 + \Delta_a s + \int_0^\infty \Delta_\nu(x)\left(e^{sx} - I\right)dx + Q \circ U(0) + \int_0^\infty (Q \circ \mu(x))e^{sx}dx,$$

where $a = (a_i)$, $\sigma^2 = (\sigma_i^2)$ and
- a_i, $\sigma_i \in \mathbb{R}$ define the Brownian motion while J_t is in phase i;
- we assume that the Lévy measures have a continuous density $\nu_i(\cdot)$ for all i, such that $\int_0^\infty \nu_i(x)dx < \infty$, as mentioned in the introduction;
- the distribution functions $U_{ij}(\cdot)$ of the jumps when J_t changes from i to j have a continuous density, except possibly at 0, so that $U_{ij}(x) = U_{ij}(0) + \int_0^x \mu_{ij}(u)du$, for all i, j, in particular, $U_{ii}(0) = 1$ and $\mu_{ii}(x) = 0$ for all $x > 0$.

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The last two assumptions are made to simplify our analysis and are not much restrictive: if the jumps should have discrete components, one would replace the Riemann integrals by Lebesgue-Stieltjes integrals.

The asymptotic drift κ plays an important role, as we see in Theorem 3.1 below. It is given by

$$(3.1) \qquad \kappa = \pi^{\mathrm{T}} F'(0) \mathbf{1} = \pi^{\mathrm{T}} \left\{ \Delta_a + \int_0^\infty x \Delta_\nu(x) dx + \int_0^\infty x(Q \circ \mu(x)) dx \right\} \mathbf{1},$$

where π is the stationary probability vector of Q: $\pi^{\mathrm{T}} Q = 0$, $\pi^{\mathrm{T}} \mathbf{1} = 1$, and it is such that $\lim_{t \to \infty} X_t = +\infty$ if $\kappa > 0$, $\lim_{t \to \infty} X_t = -\infty$ if $\kappa < 0$, and $\limsup_{t \to \infty} X_t = +\infty$, $\liminf_{t \to \infty} X_t = -\infty$ if $\kappa = 0$.

The matrix G, such that $(e^G x)_{i,j}$ is the probability that the phase is j upon the first visit to level $-x$, given that the process starts from level 0, in phase i at time 0, is characterised as follows.

THEOREM 3.1. *Assume that Q is an irreducible generator and that $\sigma_i > 0$ for all i. The matrix G is a solution of the equation $F(G) = 0$, with*

$$(3.2) \qquad \begin{aligned} F(Y) = \Delta_a Y &+ \frac{1}{2} \Delta_\sigma^2 Y^2 + \int_0^\infty \Delta_\nu(x)(e^{Yx} - I) dx \\ &+ Q \circ U(0) + \int_0^\infty (Q \circ \mu(x)) e^{Yx} dx. \end{aligned}$$

If $\kappa \leq 0$, G is a generator and is the unique solution in the set of real matrices of order n having a simple eigenvalue equal to 0 and $n-1$ eigenvalues with strictly negative real parts, if $\kappa > 0$, G is a subgenerator and is the unique solution in the set of matrices having n eigenvalues with strictly negative real parts.

Proof. Breuer [7] and D'Auria *et al.* [10] analyse Lévy processes with negative jumps and the adaptation of their results to positive jumps is immediate. As such, the claim that $F(G) = 0$ is a special case of [7, Theorem 1] and [10, Theorem 2], uniqueness is proved in [10]. ☐

REMARK 3.2. In addition to the stated properties, the matrix G is irreducible. This is a direct consequence from the assumption that Q is irreducible, and a formal argument, easy to follow, is given in Proposition 5.1.

We note for later reference that (3.2) is equivalent to

$$(3.3) \qquad \begin{aligned} F(Y) = &\frac{1}{2} \Delta_{\sigma^2} Y^2 + \Delta_a Y + \int_0^\infty \Delta_\nu(x) \left(e^{Yx} - I \right) dx \\ &+ Q + \int_0^\infty (Q \circ \mu(x))(e^{Yx} - I) dx, \end{aligned}$$

because

$$(3.4) \qquad Q \circ U(0) + \int_0^\infty (Q \circ \mu(x)) dx = Q \circ (\mathbf{1}^{\mathrm{T}} \mathbf{1}) = Q.$$

Relying on this characterization of G, in the next sections we will provide the design and analysis of algorithms for the numerical computation of G.

4. A quadratic matrix equation. We intend to connect with earlier work on algorithms for quadratic matrix equations (Bini *et al.* [5]) in this section and nonsymmetric algebraic Riccati equations (Bini *et al.* [4]) in Section 5. We set $W = I + \tau Y$, where $\tau > 0$ is fixed. Replacing Y with $\tau^{-1}(W - I)$ in (3.2) and multiplying by $2\tau^2$, yield the equivalent matrix equation in the unknown W

$$(4.1) \qquad B_1 W^2 + B_0(\tau)W + B_{-1}(\tau, W) = 0,$$

where

$$(4.2) \qquad \begin{aligned} &B_1 = \Delta_{\sigma^2}, \quad B_0(\tau) = 2(\tau\Delta_a - \Delta_{\sigma^2}), \\ &B_{-1}(\tau, W) = \Delta_{\sigma^2} - 2\tau\Delta_a + 2\tau^2 \left(H(\tau, W) + K(\tau, W) \right), \end{aligned}$$

with

$$(4.3) \qquad H(\tau, W) = \int_0^\infty \Delta_\nu(x) \left(e^{\tau^{-1}(W-I)x} - I \right) dx,$$

$$(4.4) \qquad K(\tau, W) = Q \circ U(0) + \int_0^\infty (Q \circ \mu(x)) e^{\tau^{-1}(W-I)x} dx.$$

The sign of the drift κ, defined in (3.1), determines the existence of a unique stochastic or substochastic solution of (4.1).

LEMMA 4.1. *If $\kappa \leq 0$ and (4.1) has a stochastic solution W^*, then W^* is the unique stochastic solution. If $\kappa > 0$ and (4.1) has a substochastic solution W^*, then W^* is the unique substochastic solution. In both cases, $G = \tau^{-1}(W^* - I)$.*

Proof. Assume that $\kappa \leq 0$ and that (4.1) has stochastic solutions, and let W^* be any of them. The matrix $Y^* = \tau^{-1}(W^* - I)$ is a solution of (3.2), it is a generator and, by Corollary 3.1, $Y^* = G$. In consequence, such a stochastic solution would be unique. A similar argument holds in case $\kappa > 0$. $\square$

Thus, our objective is to impose constraints on τ which guarantee that (4.1) has a stochastic or a substochastic solution. Firstly, we note that B_1 is a diagonal matrix with positive diagonal entries. Secondly, $B_0(\tau)$ is a diagonal matrix and its diagonal entries are negative if

$$(4.5) \qquad 0 < \tau < \min_{a_i > 0} \left\{ \frac{\sigma_i^2}{a_i} \right\}.$$

Finally, we need $B_{-1}(\tau, W) \geq 0$ for any (sub)stochastic matrix W, and this requires to proceed through a few steps.

Step 1: we readily conclude that the off-diagonal entries are nonnegative. This is true of the term $H(\tau, W)$ since $e^{-M} \geq 0$ for any Z-matrix M. For $K(\tau, W)$, we recall that $\mu_{ii}(x) = 0$ for all i, $Q \circ \mu(x) \geq 0$ for all x, and so the integral in (4.4) is nonnegative. Therefore, $K(\tau, W) \geq Q \circ U(0)$, and

$$\begin{aligned} K_{ij}(\tau, W) &\geq Q_{ij} U_{ij}(0) \geq 0 \qquad \text{for } i \neq j, \\ K_{ii}(\tau, W) &\geq q_{ii} \qquad \text{for all } i. \end{aligned}$$

This concludes the argument about the off-diagonal entries, and we are left with the diagonal entries of $B_{-1}(\tau, W)$.

Step 2: if Y is a (sub)generator, then e^{Yx} is (sub)stochastic for any $x \geq 0$. Hence, if W is a (sub)stochastic matrix and $Y = \tau^{-1}(W - I)$, then

$$
\begin{aligned}
K(\tau, W)\mathbf{1} &= Q \circ U(0)\mathbf{1} + \int_0^\infty (Q \circ \mu(x))e^{Yx}\mathbf{1}dx \\
&\leq Q \circ U(0)\mathbf{1} + \int_0^\infty (Q \circ \mu(x))\mathbf{1}dx \\
&= Q\mathbf{1} \qquad \text{by (3.4)} \\
&= 0,
\end{aligned}
$$

(4.6)

which implies that $K(\tau, W)$ is a (sub)generator.

Step 3: concerning the diagonals entries of $H(\tau, W)$, since $e^{\tau^{-1}x(W-I)} - I \geq -I$, we have

$$
H_{ii}(\tau, W) \geq - \int_0^\infty \nu_i(dx)
$$

and we get from (4.6)

$$
(4.7) \qquad (B_{-1}(\tau, W))_{i,i} \geq \sigma_i^2 - 2\tau a_i + 2\tau^2 \left(q_{i,i} - \int_0^\infty \nu_i(dx) \right) \qquad \text{for all } i.
$$

Since $q_{ii} < 0$, the leading coefficient of the quadratic polynomial in the right-hand side of (4.7) is negative, while the constant coefficient is positive. Therefore the polynomial has two real roots of opposite signs.

A more accurate bound may be obtained as follows: as $W \geq 0$, therefore $e^{\tau^{-1}Wx} \geq I$ and

$$
H_{ii}(\tau, W) \geq \int_0^\infty \nu_i(x)(e^{-\tau^{-1}x} - 1)dx \geq -\tau^{-1} \int_0^\infty x\nu_i(x)dx.
$$

If the latter integral is finite, (4.7) may be replaced by

$$
(4.8) \qquad (B_{-1}(\tau, W))_{i,i} \geq \sigma_i^2 - \tau \left(2a_i + \int_0^\infty x\nu_i(dx) \right) + 2\tau^2 q_{i,i} \qquad \text{for all } i.
$$

Like in (4.7), the leading coefficient of the quadratic polynomial is negative, while the constant coefficient is positive, and the polynomial has two real roots of opposite signs. Moreover, the larger the integral in (4.8), the larger the positive root of the polynomial.

We denote as τ_i the positive root of the polynomial either in (4.7) or in (4.8), and we know that $(B_{-1}(\tau, W))_{i,i} \geq 0$ if $0 < \tau < \tau_i$.

THEOREM 4.2. *Assume that W is a stochastic or substochastic matrix and that*

$$
(4.9) \qquad 0 < \tau < \tau^*, \quad \tau^* = \min \left\{ \min_{a_i > 0} \left\{ \frac{\sigma_i^2}{a_i} \right\}, \min_i \tau_i \right\}.
$$

The matrices $B_{-1}(\tau, W)$, $-B_0(\tau)$ and B_1 are nonnegative and such that $(B_{-1}(\tau, W) + B_0(\tau) + B_1)\mathbf{1} \leq 0$.

Proof. If τ satisfies (4.9), we have proved that the three matrices $B_{-1}(\tau, W)$, $-B_0(\tau)$ and B_1 are nonnegative in the discussion above. From their definition (4.2), we have

$$
\frac{1}{2}\tau^{-2}(B_{-1}(\tau, W) + B_0(\tau) + B_1) = H(\tau, W) + K(\tau, W).
$$

We know from (4.6) that $K(\tau, W)\mathbf{1} \leq 0$. As $e^{\tau^{-1}x(W-I)}\mathbf{1} \leq \mathbf{1}$, it results from (4.3) that $H(\tau, W)\mathbf{1} \leq 0$. This concludes the proof. $\qquad\square$

Assume that τ satisfies (4.9). The matrix $-B_0(\tau)$ has a nonnegative inverse and the matrices

$$(4.10) \qquad \widetilde{B}_{-1}(\tau, W) = -B_0(\tau)^{-1}B_{-1}(\tau, W), \qquad \widetilde{B}_1(\tau) = -B_0(\tau)^{-1}B_1$$

are nonnegative, with

$$(4.11) \qquad \widetilde{B}_{-1}(\tau, W) + \widetilde{B}_1(\tau)\mathbf{1} \leq \mathbf{1},$$

by Theorem 4.2. We now focus our attention on the equation

$$(4.12) \qquad W = \widetilde{B}_{-1}(\tau, W) + \widetilde{B}_1(\tau)W^2,$$

which is equivalent to (4.1).

LEMMA 4.3. *Assume that τ satisfies the inequalities (4.9) and that W_1, W_2 are (sub)stochastic matrices. If $W_1 \leq W_2$, then $\widetilde{B}_{-1}(\tau, W_1) \leq \widetilde{B}_{-1}(\tau, W_2)$.*

Proof. If W_1 and W_2 are substochastic, then $B_{-1}(\tau, W_1)$ and $B_{-1}(\tau, W_2)$ are well defined and nonnegative. The statement follows from $-B_0(\tau)^{-1} \geq 0$ and from $e^{\tau^{-1}(W_1-I)x} \leq e^{\tau^{-1}(W_2-I)x}$ for any $x \geq 0$. □

THEOREM 4.4. *Assume that τ satisfies the inequality (4.9). Define the sequence*

$$(4.13) \qquad W_{k+1} = \widetilde{B}_{-1}(\tau, W_k) + \widetilde{B}_1(\tau)W_k^2, \qquad k = 0, 1, \ldots$$

with $W_0 = 0$. Then $0 \leq W_k \leq W_{k+1}$ and $W_{k+1}\mathbf{1} \leq \mathbf{1}$ for all k. There exists $W_{\min} = \lim_{k\to\infty} W_k$. The matrix $W_{\min}$ is (sub)stochastic and is the minimal nonnegative solution to (4.1).

If $\kappa \leq 0$, $W_{\min}$ is the unique stochastic solution, if $\kappa > 0$, $W_{\min}$ is the unique substochastic solution. In both cases, $G = \tau^{-1}(W_{\min} - I)$.

Proof. The first statement is proved by induction on k. If $k = 0$, then $W_0 = 0$ and $W_1 = \widetilde{B}_{-1}(\tau, 0)$, W_1 is nonnegative by Theorem 4.2; also, (4.11) implies that $W_1\mathbf{1} \leq \mathbf{1}$. Assume now that $0 \leq W_k \leq W_{k+1}$ and $W_{k+1}\mathbf{1} \leq \mathbf{1}$ for a given $k \geq 0$. This implies that $\widetilde{B}_1(\tau)W_k^2 \leq \widetilde{B}_1(\tau)W_{k+1}^2$ and, by Lemma 4.3, that $0 \leq \widetilde{B}_{-1}(\tau, W_k) \leq \widetilde{B}_{-1}(\tau, W_{k+1})$. Therefore, $0 \leq W_{k+1} \leq W_{k+2}$. Furthermore, by the inductive assumption and by (4.11), we find that $W_{k+2}\mathbf{1} \leq \left(\widetilde{B}_{-1}(\tau, W_{k+1}) + \widetilde{B}_1(\tau)\right)\mathbf{1} \leq \mathbf{1}$.

Since the sequence $\{W_k\}_k$ is monotone non decreasing and bounded from above, there exists $W_{\min} = \lim_{k\to\infty} W_k$, solution of (4.12), and therefore of (4.1), by continuity. We show by contraddiction that $W_{\min}$ is the minimal nonnegative solution. If $0 \leq \widehat{W} \leq W_{\min}$ is another solution, one proves by induction that $W_k \leq \widehat{W}$ for any $k = 0, 1, 2\ldots$, therefore $W_{\min} = \lim_k W_k \leq \widehat{W}$, hence $W_{\min} = \widehat{W}$. The last statement directly follows from Lemma 4.1. □

5. A nonsymmetric algebraic Riccati equation. In Section 4, we replace Y by the (sub)stochastic matrix W through an artificial parameter τ, for which we determine the constraint (4.9). In this section, we decompose G as the sum

$$(5.1) \qquad G = -\Delta_b + \Delta_b\Psi$$

where Δ_b and Ψ have a physical meaning. This is inspired from a similar decomposition for fluid queues, see Latouche and Nguyen [19, Equations (13) and (15)].

If the process starts from phase i at time 0, it behaves like a Brownian motion with parameters a_i and σ_i until the random time T where either there is a change of

phase (the rate is $|q_{ii}|$) or there is a jump in the level (the rate is $\rho_i = \int_0^\infty \nu_i(z)\,dz$). That random variable T is exponentially distributed, with parameter $\lambda_i = \rho_i + |q_{ii}|$. We condition on the value of the level immediately after time T.

It is known (Sato [27, Section 45]) that over an exponential interval, the minimum $m(T) = \min\{X(s) : 0 \le s \le T\}$ and the difference $X(T) - m(T)$ are independent and exponential random variables, with respective parameters

$$(5.2) \qquad b_i = (\sqrt{a_i^2 + 2\lambda_i \sigma_i^2} + a_i)/\sigma_i^2 \quad \text{and} \quad c_i = b_i - 2a_i/\sigma_i^2, \qquad i = 1, \ldots, n.$$

We analyze the trajectory of the process over the interval $[0, T]$, and find

$$
e^{Gx} = e^{-\Delta_b x} + \int_0^x \Delta_b e^{-\Delta_b v}\,dv \int_0^\infty \Delta_c e^{-\Delta_c y}\,dy \times
$$

$$(5.3) \qquad \left\{ \int_0^\infty \operatorname{diag}(\widehat{P})\Delta_f(z) e^{G(x+y+z-v)}\,dz + ((\widehat{P} - \operatorname{diag}(\widehat{P})) \circ U(0))\, e^{G(x+y-v)} \right.$$

$$\left. + \int_0^\infty ((\widehat{P} - \operatorname{diag}(\widehat{P})) \circ \mu(z)) e^{G(x+y+z-v)}\,dz \right\}$$

where $\widehat{P} = \Delta_\lambda^{-1} Q + I$ is the phase transition matrix at time T, and $f_i(z) = \nu_i(z)/\rho_i$ is the density of the jumps if there is no change of phase.

The justification of (5.3) goes as follows, recall that the exponential on the left is the transition matrix from level 0 to level $-x$.

- The first term is the probability that $|m(T)| > x$. In that case, the level has reached $-x$ before there is any change of phase.
- The double integral is the probability that $m(T) = -v$ with $0 < v < x$, and that $X(T) - m(T) = y$ with $y > 0$.
- The expression in brackets is the probability that at time T there is a jump of size $z \ge 0$, so that the level has to go down by a total of $x - v + y + z$ units.

We take the derivative with respect to x, evaluated at 0, and obtain (5.1), where

$$
\Psi = \int_0^\infty \Delta_c e^{-\Delta_c y}\,dy \left\{ \int_0^\infty \operatorname{diag}(\widehat{P})\Delta_f(z)\,dz\, e^{G(y+z)} \right.
$$

$$(5.4) \qquad \left. + (\widehat{P} - \operatorname{diag}(\widehat{P})) \circ U(0)\, e^{Gy} + \int_0^\infty ((\widehat{P} - \operatorname{diag}(\widehat{P})) \circ \mu(z))\,dz\, e^{G(y+z)} \right\},
$$

and $b = (b_i)_i$ and $c = (c_i)_i$ are defined in (5.2). After some algebraic manipulation, (5.4) is shown to be equivalent to (3.2) with the substitution (5.1).

PROPOSITION 5.1. *If the matrix Q is irreducible, then the matrices Ψ, G and $W_{\min}$ are irreducible.*

Proof. The proof is adapted from the argument in Van Lierde *et al.* [31, Lemma 3.2]. By construction, Ψ is a (sub)stochastic matrix: its entry Ψ_{ij} is the probability that, starting at time 0 from level 0 in phase i with a jump to level $y + z$, the process is in phase j upon return to level 0.

As the transition matrix Q is irreducible, there exists a path $i = i_1, i_2, \ldots, i_\ell = j$ with $Q_{i_k, i_{k+1}} > 0$ for $k = 1, \ldots, \ell - 1$, and there is a strictly positive probability that the process moves through the phases i_1 to i_ℓ on its trajectory from level $y + z$ to level 0. Therefore, $\Psi_{ij} > 0$ for all i and j, and it follows from Equation (5.1) and Theorem 4.4, respectively, that G and $W_{\min}$ are irreducible as well. □

We replace G with its expression (5.1) in $F(G) = 0$ and, using (5.2), obtain

(5.5)
$$0 = \frac{1}{2}\Delta_\sigma^2\Delta_b\Psi\Delta_b\Psi - \frac{1}{2}\Delta_\sigma^2(\Delta_b\Delta_c\Psi + \Delta_b\Psi\Delta_b) + \frac{1}{2}\Delta_\sigma^2\Delta_b\Delta_c$$
$$+ Q\circ U(0) + \int_0^\infty \Delta_\nu(x)(e^{Gx} - I)dx + \int_0^\infty (Q\circ\mu(x))e^{Gx}\,dx.$$

We readily see that

(5.6)
$$\frac{1}{2}\Delta_\sigma^2\Delta_b\Delta_c = \Delta_\lambda = \Delta_\rho - \Delta_q$$

where $\Delta_q = \mathrm{diag}(q_{ii})$ (recall that $U_{ii}(0) = 1$ for all i), so that (5.5) becomes

$$0 = \frac{1}{2}\Delta_\sigma^2\Delta_b\left\{\Psi\Delta_b\Psi - \Delta_c\Psi - \Psi\Delta_b\right\} + C(\Psi)$$

where

$$C(\Psi) = (Q - \Delta_q)\circ U(0) + \int_0^\infty \Delta_\nu(x)\,e^{(\Delta_b\Psi - \Delta_b)x}dx + \int_0^\infty (Q\circ\mu(x))e^{(\Delta_b\Psi - \Delta_b)x}\,dx.$$

The matrix $C(\Psi)$ is non-negative because both $Q - \Delta_q$ and $Q\circ\mu(x)$ have zeros on the diagonal, and non-negative off-diagonal elements.

To prove the next lemma, we follow the same argument as for Lemma 4.1.

LEMMA 5.2. *If $\kappa \leq 0$, the matrix equation*

$$0 = \frac{1}{2}\Delta_\sigma^2\Delta_b\left\{X\Delta_b X - \Delta_c X - X\Delta_b\right\} + C(X)$$

has a unique stochastic solution Ψ; if $\kappa > 0$, the equation has a unique substochastic solution Ψ. In both cases, $G = -\Delta_b + \Delta_b\Psi$. □

The matrix $\Delta_b\Psi$ solves the equation

(5.7)
$$X^2 - \Delta_c X - X\Delta_b + 2\Delta_\sigma^{-2}\widehat{C}(X) = 0$$

in the unknown X, where

(5.8) $\widehat{C}(X) = (Q - \Delta_q)\circ U(0) + \int_0^\infty \Delta_\nu(x)\,e^{(X-\Delta_b)x}dx + \int_0^\infty (Q\circ\mu(x))e^{(X-\Delta_b)x}\,dx.$

If W is a given matrix and if we interpret $\widehat{C}(W)$ as a constant coefficient, equation (5.7) can be transformed into a nonsymmetric algebraic Riccati equation (NARE)

(5.9)
$$S^2 - \Delta_c S - S\Delta_b + 2\Delta_\sigma^{-2}\widehat{C}(W) = 0.$$

PROPOSITION 5.3. *If $W \geq 0$ and $W\mathbf{1} \leq b$, then the matrix*

$$M = \begin{bmatrix} \Delta_b & -I \\ -2\Delta_\sigma^{-2}\widehat{C}(W) & \Delta_c \end{bmatrix}$$

is an irreducible M-matrix and it is singular if $W\mathbf{1} = b$. Equation (5.9) has a minimal nonnegative solution $S_{\min}(W) > 0$ and $S_{\min}(W)\mathbf{1} \leq b$. Finally, $S_{\min}(W)\mathbf{1} = b$ if $W\mathbf{1} = b$.

Proof. As $W \geq 0$ and $W\mathbf{1} \leq b$, $W - \Delta_b$ is a generator and $e^{(W-\Delta_b)x}$ is a nonnegative (sub)stochastic matrix with strictly positive diagonal. This implies that $\widehat{C}(W)$ is irreducible nonnegative, and M is, therefore, an irreducible Z-matrix. We show next that

$$\widetilde{M} = M \begin{bmatrix} I & 0 \\ 0 & \Delta_b \end{bmatrix} = \begin{bmatrix} \Delta_b & -\Delta_b \\ -2\Delta_\sigma^{-2}\widehat{C}(W) & \Delta_c\Delta_b \end{bmatrix}$$

satisfies the property $\widetilde{M}\mathbf{1} \geq 0$, and $\widetilde{M}\mathbf{1} = 0$ if $W\mathbf{1} = b$. Set $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \widetilde{M}\begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix}$. Clearly $y_1 = 0$, and we only need to deal with y_2. As $e^{(W-\Delta_b)t}$ is substochastic,

(5.10)
$$\widehat{C}(W)\mathbf{1} \leq (Q - \Delta_q) \circ U(0)\mathbf{1} + \int_0^\infty \Delta_\nu(dx)\,\mathbf{1} + \int_0^\infty (Q \circ \mu(x))\,dx\mathbf{1}$$
$$= Q\mathbf{1} + \Delta_\lambda\mathbf{1} = \Delta_\lambda\mathbf{1},$$

and $y_2 \geq -2\Delta_\sigma^{-2}\Delta_\lambda\mathbf{1} + \Delta_c\Delta_b\mathbf{1} = 0$. If $W\mathbf{1} = b$, then $e^{(W-\Delta_b)t}$ is stochastic, so that $\widehat{C}(W)\mathbf{1} = \Delta_\lambda\mathbf{1}$ and $y_2 = 0$. This implies by Theorem 2.1 that M is an (irreducible) M-matrix, singular if $W\mathbf{1} = b$ and, by Bini *et al.* [4, Theorem 2.9], that there exists a minimal nonnegative solution which is strictly positive.

Now we show that $S(W)\mathbf{1} \leq b$. By the argument in the proof of [4, Theorem 2.9], $S_{\min}(W) = \lim_{k\to\infty} S_k(W)$, where $S_0(W) = 0$ and

(5.11)
$$\Delta_c S_{k+1}(W) + S_{k+1}(W)\Delta_b = S_k(W)^2 + 2\Delta_\sigma^{-2}\widehat{C}(W), \quad k = 0, 1, \dots$$

Moreover, $0 \leq S_k(W) \leq S_{k+1}(W)$. We prove by induction on k, that $S_k(W)\mathbf{1} \leq b$ for any $k \geq 0$, so that in the limit as $k \to \infty$, $S_{\min}(W)\mathbf{1} \leq b$. If $k = 0$, $S_0(W) = 0$ and the inequality holds. Assume that $S_k(W)\mathbf{1} \leq b$. By monotonicity of the sequence $\{S_k(W)\}$ we have

$$\Delta_c S_{k+1}(W)\mathbf{1} + S_{k+1}(W)\Delta_b\mathbf{1} \leq S_k(W)^2\mathbf{1} + \Delta_c\Delta_b\mathbf{1} \qquad \text{by (5.6, 5.10)}$$
$$\text{or} \quad \Delta_c S_{k+1}(W)\mathbf{1} \leq -S_{k+1}(W)b + S_k(W)b + \Delta_c b,$$

that is, $S_{k+1}(W)\mathbf{1} \leq b$. Finally, if $W\mathbf{1} = b$, we have seen that $\widehat{C}(W)\mathbf{1} = \Delta_\lambda\mathbf{1}$ and we obtain from (5.9) that

$$0 = S_{\min}(W)^2\mathbf{1} - \Delta_c S_{\min}(W)\mathbf{1} - S_{\min}(W)b + 2\Delta_\sigma^2\Delta_\lambda\mathbf{1}$$
$$= (S_{\min}(W) - \Delta_c)(S_{\min}(W)\mathbf{1} - b) = S_{\min}(W)\mathbf{1} - b$$

since $S_{\min}(W) - \Delta_c$ is nonsingular by [5, Theorem 2.9]. This completes the proof. $\square$

LEMMA 5.4. *If* $0 \leq X \leq Y$, $X\mathbf{1} \leq b$ *and* $Y\mathbf{1} \leq b$, *then* $\widehat{C}(X) \leq \widehat{C}(Y)$ *and* $S_{\min}(X) \leq S_{\min}(Y)$.

Proof. From (2.1),

$$e^{(Y-\Delta_b)x} - e^{(X-\Delta_b)x} = \int_0^x e^{(X-\Delta_b)(x-s)}(Y-X)e^{(Y-\Delta_b)s}ds \geq 0$$

since $Y - X \geq 0$ and $\Delta_b - X$, $\Delta_b - Y$ are M-matrices. Therefore, from (5.8), we have $\widehat{C}(X) \leq \widehat{C}(Y)$.

To prove the second statement, we recall that $S_{\min}(W) = \lim_{k\to\infty} S_k(W)$, where $S_k(W)$ is defined in (5.11), with $S_0(W) = 0$. We may easily show by induction on k that $S_k(X) \leq S_k(Y)$, which implies that $S_{\min}(X) \leq S_{\min}(Y)$. $\square$

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The next theorem is a direct consequence of Lemma 5.2, Proposition 5.3 and (5.1).

THEOREM 5.5. *If $W = \Delta_b \Psi$, then the minimal nonnegative solution of (5.9) is $S_{\min}(W) = \Delta_b \Psi$. Moreover, $\Psi > 0$ and the off-diagonal entries of G are strictly positive.* □

6. Computational methods. The recurrence (4.13), with $W_0 = 0$, provides a fixed point iteration for computing the minimal nonnegative solution to (4.1). Its convergence, however, may be slow and we develop faster algorithms in this section.

6.1. U-based iteration. We introduce a fixed point iteration which resembles the U-based iteration for solving matrix equations arising in M/G/1-type Markov chains (Bini *et al.* [5, Section 6.2]). Take $W_0 = 0$ and define

$$(6.1) \qquad W_{k+1} = (I - \widetilde{B}_1(\tau)W_k)^{-1}\widetilde{B}_{-1}(\tau, W_k), \;\; k = 0, 1, \dots,$$

where the matrix $I - \widetilde{B}_1(\tau)W_k$ is invertible, as well as $I - \widetilde{B}_1(\tau)W_{\min}$, in view of the arguments in [5, Section 6.2]. By the same arguments as in the proof of Theorem 4.4, one may show that the sequence W_k converges monotonically to the minimal nonnegative solution $W_{\min}$ of (4.1).

A pseudocode of this iteration is reported in Algorithm 6.1.

Algorithm 6.1 U-based algorithm

Input: The parameters $a, \sigma, \nu, \mu, Q, U$ defining (3.2), a value of τ satisfying (4.9), a
    starting approximation W_0, and a tolerance $\epsilon > 0$.
Output: An approximation G to the solution of (3.2).
 1: Compute $W_1 = (I - \widetilde{B}_1(\tau)W_0)^{-1}\widetilde{B}_{-1}(\tau, W_0)$, where $\widetilde{B}_1(\tau)$ and $\widetilde{B}_{-1}(\tau, W_0)$ are
    defined in (4.10).
 2: Compute $\delta = \|W_1 - W_0\|_\infty$ and set $k = 1$.
 3: **while** $\delta > \epsilon$ **do**
 4:    Compute $W_{k+1} = (I - \widetilde{B}_1(\tau)W_k)^{-1}\widetilde{B}_{-1}(\tau, W_k)$.
 5:    Compute $\delta = \|W_{k+1} - W_k\|_\infty$ and set $k = k + 1$.
 6: **end while**
 7: Set $G = \tau^{-1}(W_k - I)$.

We provide a convergence analysis for the sequence (6.1), a similar analysis may be repeated for the sequence (4.13). Define the error at step k as

$$E_k = W_{\min} - W_k, \;\; k = 0, 1, \dots$$

As $W_k \le W_{\min}$, we have $E_k \ge 0$ for any $k \ge 0$ and, in particular, $\|E_k\|_\infty = \|E_k \mathbf{1}\|_\infty$. By subtracting (6.1) from

$$W_{\min} = (I - \widetilde{B}_1(\tau)W_{\min})^{-1}\widetilde{B}_{-1}(\tau, W_{\min})$$

we find that

$$(6.2) \qquad E_{k+1} = (I - \widetilde{B}_1(\tau)W_{\min})^{-1}\left(-2\tau^2 B_0(\tau)^{-1}A_k + \widetilde{B}_1(\tau)E_k W_{k+1}\right),$$

where

$$A_k = H(W_{\min}, \tau) - H(W_k, \tau) + K(W_{\min}, \tau) - K(W_k, \tau)$$

$$(6.3) \qquad = \int_0^\infty \Delta_\nu(x)\Gamma_k(x)\,dx + \int_0^\infty (Q \circ \mu(x))\Gamma_k(x)\,dx$$

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by (4.2), with

$$\Gamma_k(x) = e^{\tau^{-1}x(W_{\min}-I)} - e^{\tau^{-1}x(W_k-I)} = \int_0^{\tau^{-1}x} e^{(W_k-I)(\tau^{-1}x-s)} E_k e^{(W_{\min}-I)s} ds$$

by (2.1). The integrand is nonnegative since $E_k \geq 0$, and $e^{(W_{\min}-I)s}\mathbf{1} \leq \mathbf{1}$ since $W_{\min}\mathbf{1} \leq \mathbf{1}$. Therefore,

$$\Gamma_k(x)\mathbf{1} \leq \left(\int_0^{\tau^{-1}x} e^{(W_k-I)(\tau^{-1}x-s)} ds \right) E_k\mathbf{1} \leq \left(\lim_{k\to\infty} \int_0^{\tau^{-1}x} e^{(W_k-I)(\tau^{-1}x-s)} ds \right) E_k\mathbf{1}.$$

Since $\{W_k\}_k$ converges monotonically to $W_{\min}$, then we get

$$(6.4) \qquad \Gamma_k(x)\mathbf{1} \leq \int_0^{\tau^{-1}x} e^{(W_{\min}-I)(\tau^{-1}x-s)} ds\, E_k\mathbf{1} = \tau^{-1}\int_0^x e^{Gs} ds\, E_k\mathbf{1}$$

by Theorem 4.4. We combine (6.3, 6.4) and write

$$(6.5) \qquad\qquad\qquad A_k\mathbf{1} \leq \tau^{-1}\Lambda E_k\mathbf{1},$$

where

$$(6.6) \qquad \Lambda = \int_0^\infty \Delta_\nu(x) \int_0^x e^{Gs}\, ds\, dx + \int_0^\infty (Q \circ \mu(x)) \int_0^x e^{Gs}\, ds\, dx.$$

Hence, from (6.2),

$$E_{k+1}\mathbf{1} \leq RE_k\mathbf{1} \leq R^{k+1}E_0\mathbf{1}$$

where

$$R = (I - \widetilde{B}_1(\tau)W_{\min})^{-1}\left(-2\tau B_0(\tau)^{-1}\Lambda + \widetilde{B}_1(\tau) \right),$$

and so $\|E_k\mathbf{1}\| \leq \|R^k\|\|E_0\mathbf{1}\|$ for any vector norm $\|\cdot\|$ where $\|A\|$ is the corresponding induced matrix norm. Therefore the asymptotic rate of convergence, given by $\lim_k \|R^k\|^{1/k}$ coincides with $\rho(R)$, and our objective in the remainder of this subsection is to further analyse R.

We replace $\widetilde{B}_1(\tau)$ with its expression (4.10), and obtain

$$(6.7) \qquad\qquad R = -(B_0(\tau) + B_1(\tau G + I))^{-1}(2\tau\Lambda + B_1)$$

next, we replace $B_0(\tau)$ and B_1 with their expressions in (4.2), and obtain

$$R = (\tau^{-1}\Delta_\sigma^2 - 2\Delta_a - \Delta_\sigma^2 G)^{-1}(2\Lambda + \tau^{-1}\Delta_\sigma^2).$$

We see that $R = M_1^{-1}N_1$, with

$$(6.8) \qquad M_1 = \tau^{-1}\Delta_\sigma^2 - 2\Delta_a - \Delta_\sigma^2 G \quad \text{and} \quad N_1 = 2\Lambda + \tau^{-1}\Delta_\sigma^2.$$

Moreover, $M_1 - N_1$ is equal to the matrix

$$(6.9) \qquad\qquad\qquad \Theta = -2\Delta_a - \Delta_\sigma^2 G - 2\Lambda.$$

We show in Theorem 6.1 below that Θ is an irreducible M-matrix and that $\Theta = M_1 - N_1$ is a regular splitting. Moreover if $\kappa < 0$ then Θ is nonsingular, so that $\rho(R) < 1$ by Theorem 2.2. Furthermore, since N_1 is a nonincreasing function of τ, the spectral radius of R is a nonincreasing function of τ, and the larger τ, the faster the convergence, provided that τ satisfies condition (4.5).

THEOREM 6.1. *The matrix Θ is irreducible. If $\kappa < 0$, then it is a nonsingular M-matrix. Otherwise, if $\kappa \geq 0$, then it is a singular M-matrix. Moreover, $\Theta = M_1 - N_1$ is a regular splitting, where M_1, N_1 are defined in* (6.8).

Proof. The matrix Λ is nonnegative and G is an irreducible (sub)generator with strictly positive off-diagonal entries in view of Theorem 5.5. Therefore, the off-diagonal entries of Θ are strictly negative, that is, Θ is an irreducible Z-matrix. With $\pi > 0$ the stationary probability vector of Q, we only need to prove that $\pi^T \Theta > 0$ if $\kappa < 0$ and that $\pi^T \Theta = 0$ if $\kappa \geq 0$, so that, by Theorem 2.1, Θ is a nonsingular M-matrix in the first case, a singular M-matrix in the second.

We observe that $\int_0^x e^{Gs} G \, ds = e^{Gx} - I$, and so

$$\Theta G = -2\Big\{\Delta_a G + \frac{1}{2}\Delta_\sigma^2 G^2 + \int_0^\infty \Delta_\nu(x)(e^{Gx} - I)dx$$

$$+ \int_0^\infty (Q \circ \mu(x))(e^{Gx} - I)dx\Big\} = -2Q,$$

using $F(G) = 0$ and (3.3). This implies that

$$\text{(6.10)} \qquad\qquad\qquad \pi^T \Theta G = 0.$$

If $\kappa > 0$, then G is a subgenerator, it is nonsingular and we find from (6.10) that $\pi^T \Theta = 0$. If $\kappa \leq 0$, then G is a generator and

$$\Theta \mathbf{1} = -2(\Delta_a \mathbf{1} + \Lambda \mathbf{1}) = -2\Big\{\Delta_a \mathbf{1} + \int_0^\infty x\Delta_\nu(x)dx + \int_0^\infty (Q \circ \mu(x))dx\Big\}$$

which implies that $\pi^T \Theta \mathbf{1} = -2\kappa$ by (3.1) and Corollary 3.1. Also, by (6.10), we have

$$\text{either} \qquad \pi^T \Theta = 0 \qquad \text{or} \qquad \pi^T \Theta = \gamma s^T, \quad \gamma \neq 0,$$

where s is the steady state vector of G, that is, $s^T G = 0$, $s^T \mathbf{1} = 1$ (recall from Proposition 5.1 that G is irreducible). Now, if $\kappa < 0$, then $\pi^T \mathbf{1}$ cannot be 0, and we are left with the option $\pi^T = \gamma s^T$ with $\gamma > 0$, so that $\pi^T \Theta > 0$. Finally, if $\kappa = 0$, then $\pi^T \Theta$ cannot be equal to γs^T with $\gamma \neq 0$, the only possibility is $\pi^T \Theta = 0$.

Concerning the last claim, the matrix N_1 is clearly nonnegative, the matrix M_1 is a Z-matrix and is such that $M_1 \geq \Theta$. Therefore, in view of Theorem 2.1, M_1 is an M-matrix. M_1 is nonsingular since its diagonal entries are strictly larger than the corresponding diagonal entries of Θ. Hence $M_1^{-1} \geq 0$. $\square$

6.2. QME-based iteration. Here we propose a more effective algorithm, that consists in generating a sequence of matrices $\{W_k\}_k$, such that W_{k+1} is the minimal nonnegative solution of the quadratic matrix equation

$$\text{(6.11)} \qquad\qquad W_{k+1} = \widetilde{B}_{-1}(\tau, W_k) + \widetilde{B}_1(\tau)W_{k+1}^2, \quad k = 0, 1, \dots$$

with $W_0 = 0$. By following the arguments of Theorem 4.4, one shows that the sequence $\{W_k\}_k$ monotonically converges to $W_{\min}$; we do not repeat the details.

A pseudocode of this iteration is reported in Algorithm 6.2. The computation of W_{k+1}, given W_k, at steps 1 and 4 can be performed by using fixed point iterations or cyclic reduction [5].

Now we analyse the speed of convergence. Define the error $E_k = W_{\min} - W_k$. By subtracting (6.11) from (4.12), we find that

$$(I - \widetilde{B}_1(\tau)W_{\min})E_{k+1} - \widetilde{B}_1(\tau)E_{k+1}W_{k+1} = -2\tau^2 B_0(\tau)^{-1}A_k,$$

Algorithm 6.2 QME-based algorithm

Input: The parameters $a, \sigma, \nu, \mu, Q, U$ defining (3.2), a value of τ satisfying (4.9), a starting approximation W_0, and a tolerance $\epsilon > 0$.

Output: An approximation G to the solution of (3.2)

  1: Compute W_1 such that $W_1 = \widetilde{B}_{-1}(\tau, W_0) + \widetilde{B}_1(\tau) W_1^2$, where $\widetilde{B}_1(\tau)$ and $\widetilde{B}_{-1}(\tau, W_0)$ are defined in (4.10).

  2: Compute $\delta = \|W_1 - W_0\|_\infty$ and set $k = 1$.

  3: **while** $\delta > \epsilon$ **do**

  4:     Compute W_{k+1} such that $W_{k+1} = \widetilde{B}_{-1}(\tau, W_k) + \widetilde{B}_1(\tau) W_{k+1}^2$.

  5:     Compute $\delta = \|W_{k+1} - W_k\|_\infty$ and set $k = k + 1$.

  6: **end while**

  7: Set $G = \tau^{-1}(W_k - I)$.

---

where A_k is defined in (6.3). By setting $P = (I - \widetilde{B}_1(\tau) W_{\min})^{-1} \widetilde{B}_1(\tau)$ and $T_k = -2\tau^2 (I - \widetilde{B}_1(\tau) W_{\min})^{-1} B_0(\tau)^{-1} A_k$, we successively get

$$E_{k+1} = \sum_{j=0}^{\infty} P^j T_k W_{k+1}^j,$$

$$
\begin{aligned}
E_{k+1} \mathbf{1} &\leq \sum_{j=0}^{\infty} P^j T_k \mathbf{1} = (I - P)^{-1} T_k \mathbf{1} \\
&= -2\tau^2 (I - P)^{-1} (I - \widetilde{B}_1(\tau) W_{\min})^{-1} B_0(\tau)^{-1} A_k \mathbf{1} \\
&= -2\tau^2 (B_0(\tau) + B_1 W_{\min} + B_1)^{-1} A_k \mathbf{1} \\
&= -2\tau (2\Delta_a + \Delta_\sigma^2 G)^{-1} A_k \mathbf{1} \qquad \text{by (4.2) and Theorem 4.4,} \\
&\leq -2(2\Delta_a + \Delta_\sigma^2 G)^{-1} \Lambda E_k \mathbf{1},
\end{aligned}
$$

where Λ is defined in (6.6), the last inequality being shown by the same argument as led us to (6.5). Therefore,

$$(6.12) \qquad E_{k+1} \mathbf{1} \leq \widehat{R} E_k \mathbf{1}, \quad \text{with} \quad \widehat{R} = -2(2\Delta_a + \Delta_\sigma^2 G)^{-1} \Lambda.$$

Observe that the matrix $\widehat{R}$ is independent of τ and that $\widehat{R} = M_2^{-1} N_2$, with

$$(6.13) \qquad M_2 = -2\Delta_a - \Delta_\sigma^2 G \quad \text{and} \quad N_2 = 2\Lambda,$$

two matrices such that $M_2^{-1} \geq 0$, $N_2 \geq 0$ and $M_2 - N_2 = \Theta$ defined in (6.9). That is, $M_2 - N_2$ is a regular splitting of Θ, different from the splitting $M_1 - N_1$ of (6.8). Since $N_2 \leq N_1$ for any $\tau > 0$, $\rho(\widehat{R}) \leq \rho(R)$ by Theorem 2.2. The inequality is strict if $\widehat{R}$ is irreducible for the Perron-Frobenius theorem [2] or if $\Theta^{-1} > 0$ (see Theorem 2.2). As a matter of fact, $\lim_{\tau \to \infty} R = \widehat{R}$ but τ must satisfy the condition (4.5), and so the speed of convergence of the iteration (6.1) cannot reach the speed of convergence of (6.11).

6.3. NARE-based iteration. Set $S_0 = 0$ and define the sequence $\{S_k\}$, where S_{k+1} is the minimal nonnegative solution of the NARE

$$(6.14) \qquad S_{k+1}^2 - \Delta_c S_{k+1} - S_{k+1} \Delta_b + 2\Delta_\sigma^{-2} \widehat{C}(S_k) = 0, \qquad k = 0, 1, \dots$$

This sequence is well defined and converges to the minimal nonnegative solution $\Delta_b \Psi$ of (5.7), as stated in the next theorem.

THEOREM 6.2. *If $S_0 = 0$, then* (6.14) *has a minimal nonnegative solution S_{k+1} for any $k \geq 0$. Moreover $S_k \mathbf{1} \leq b$, $0 \leq S_k \leq S_{k+1}$ and $\lim_k S_k = S_{\min}$, where $S_{\min} = \Delta_b \Psi$ is the minimal nonnegative solution of* (5.7) *among the nonnegative solutions such that $S\mathbf{1} \leq b$.*

Proof. The proof is by induction on k. For $k = 0$, as $S_0 \geq 0$ and $S_0 \mathbf{1} \leq b$, (6.14) has by Proposition 5.3 a minimal nonnegative solution $S_1 \geq S_0$ such that $S_1 \mathbf{1} \leq b$. Assume that $0 \leq S_{k-1} \leq S_k$ and $S_k \mathbf{1} \leq b$ for a given k. By Proposition 5.3 again, (6.14) has a minimal nonnegative solution $S_{k+1} = S(S_k)$ such that $S_{k+1} \mathbf{1} \leq b$ and by Lemma 5.4, $0 \leq S_{k-1} \leq S_k$ implies that $S_{\min}(S_{k-1}) \leq S_{\min}(S_k)$, that is, $S_k \leq S_{k+1}$.

The sequence $\{S_k\}_k$ is monotone and bounded, there exists, therefore, a limit $S^* = \lim_{k\to\infty} S_k$ and S^* solves (5.7) by continuity of $C(X)$. If there exists another solution $Y \geq 0$ such that $Y\mathbf{1} \leq b$, then one proves by induction that $S_k \leq Y$ for all k. Therefore $S^* \leq Y$, $S^* = S_{\min}$ and $S_{\min} = \Delta_b \Psi$ by Theorem 5.5. This concludes the proof. □

A pseudocode of this iteration is reported in Algorithm 6.3. The computation of S_{k+1}, given S_k, at steps 1 and 4 can be performed by using fixed point iterations or the Structured Doubling Algorithm [4].

Algorithm 6.3 NARE-based algorithm

Input: The parameters $a, \sigma, \nu, \mu, Q, U$ defining (3.2), a starting approximation S_0, a tolerance $\epsilon > 0$.
Output: An approximation G to the solution of (3.2)
1: Compute S_1 such that $S_1^2 - \Delta_c S_1 - S_1 \Delta_b + 2\Delta_\sigma^{-2} \widehat{C}(S_0) = 0$, where b and c are defined in (5.2) and $\widehat{C}(\cdot)$ is defined in (5.8).
2: Compute $\delta = \|S_1 - S_0\|_\infty$ and set $k = 1$.
3: **while** $\delta > \epsilon$ **do**
4:     Compute S_{k+1} such that $S_{k+1}^2 - \Delta_c S_{k+1} - S_{k+1} \Delta_b + 2\Delta_\sigma^{-2} \widehat{C}(S_k) = 0$.
5:     Compute $\delta = \|S_{k+1} - S_k\|_\infty$ and set $k = k + 1$.
6: **end while**
7: Set $G = S_k - \Delta_b$.

We perform a convergence analysis of the sequence $\{S_k\}_k$. Define the error $\widehat{E}_k = \Delta_b \Psi - S_k$. Since $\widehat{E}_k \geq 0$ by the monotonic convergence, we use $\widehat{E}_k \mathbf{1}$ as a measure of the error. By arguments similar to the ones in the previous sections, we find that

$$(\Delta_c - S_{k+1})\widehat{E}_{k+1} - \widehat{E}_{k+1} G = 2\Delta_\sigma^{-2}(\widehat{C}(\Delta_b \Psi) - \widehat{C}(S_k)).$$

Next, we repeat the argument between (6.3) and (6.5) and show that

$$(\widehat{C}(\Delta_b \Psi) - \widehat{C}(S_k))\mathbf{1} \leq \Lambda \widehat{E}_k \mathbf{1}$$

where Λ is defined in (6.6). As G is a (sub)generator, $G\mathbf{1} \leq 0$ and

$$(6.15) \qquad\qquad (\Delta_c - S_{k+1})\widehat{E}_{k+1}\mathbf{1} \leq 2\Delta_\sigma^{-2}\Lambda \widehat{E}_k \mathbf{1}.$$

Observe that

$$\Delta_c - S_{k+1} \geq \Delta_c - \Delta_b \Psi = \Delta_c - \Delta_b - G = -(G + 2\Delta_\sigma^{-2}\Delta_a) = \Delta_\sigma^{-2} M_2$$

with M_2 defined in (6.13). Therefore, we obtain from (6.15) that

$$M_2 \widehat{E}_{k+1}\mathbf{1} \leq 2\Delta_\sigma^{-2}\Lambda \widehat{E}_k \mathbf{1}$$

and, since M_2 has a nonnegative inverse,

$$\widehat{E}_{k+1}\mathbf{1} \le -2(\Delta_\sigma^2 Y + 2\Delta_a)^{-1}\Lambda\widehat{E}_k\mathbf{1},$$

that is, the same inequality as (6.12). Hence, the NARE-based iteration has the same asymptotic speed of convergence as the QME-based iteration.

REMARK 6.3. Define

$$\mathcal{G}(M) = \int_0^\infty \Delta_\nu(x)\left(e^{Mx} - I\right) dx + Q \circ U(0) + \int_0^\infty (Q \circ \mu(x))e^{Mx}dx.$$

The matrix G is the solution of

$$\Delta_{\sigma^2} Y^2 + 2\Delta_a Y + 2\mathcal{G}(Y) = 0$$

with characteristics given in Corollary 3.1. By simple but tedious manipulations, we may show that $W_{k+1} = I + \tau X$ with

$$\Delta_{\sigma^2} X^2 + 2\Delta_a X + 2\mathcal{G}(\tau^{-1}(W_k - I)) = 0,$$

and also that $S_{k+1} = \Delta_b + Z$, with

$$\Delta_{\sigma^2} Z^2 + 2\Delta_a Z + 2\mathcal{G}(S_k - \Delta_b) = 0.$$

In view of these similar equations, it is not surprising that the QME- and the NARE-based iterations should have the same asymptotic rate of convergence. Moreover, if S_0 and W_0 are such that $S_0 - \Delta_b = \tau^{-1}(W_0 - I)$ then for any $k \ge 0$ we have $S_k - \Delta_b = \tau^{-1}(W_k - I)$. That is, the approximation to G obtained by the NARE-based iteration coincides with the approximation obtained by the QME-based iteration.

7. Numerical experiments. We have implemented the U-, QME- and NARE-based algorithms 6.1, 6.2 and 6.3 in Matlab. As threshold value for the stopping condition we have chosen $\epsilon = 10^{-14}$. The QME and the NARE at steps 1, 4 of Algorithms 6.1 and 6.3, respectively, have been solved by means of Cyclic Reduction (CR) and Structured Doubling Algorithm (SDA); we have used the Matlab functions `cr` and `sda` from the book [4], downloadable from http://riccati.dm.unipi.it/nsare/mfiles.html. The integrals defining the matrices $\widetilde{B}_{-1}(\tau, W)$ in Algorithms 6.1 and 6.2, and $\widehat{C}(X)$ in Algorithm 6.3, have been computed by means of the Matlab function `integral`, with precision `'RelTol',1.e-10, 'AbsTol',1.e-12`.

The experiments have been run on a PC with CPU Intel Core i7-7700, under Ubuntu system, with Matlab version R2018a.

We have compared our algorithms with the algorithms proposed by Simon in [30] and by Breuer in [7]. The iteration in Simon [30, Proposition 4.8] is

$$S_{k+1}(S_k - \Delta_b) - \Delta_c S_{k+1} + 2\Delta_\sigma^{-2}\widehat{C}(S_k) = 0, \qquad k = 0, 1, \dots$$

with $S_0 = 0$.

The Breuer algorithm consists in a preprocessing stage, where the minimal solution s_i of the equation

$$a_i s + \frac{1}{2}\sigma_i^2 s^2 + \int_0^\infty (e^{sx} - 1)\nu_i(x)dx = |q_{ii}|, \quad i = 1, \dots, n$$

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is computed. The iteration, with $G_0 = 0$, is defined by

$$e_i^{\mathrm{T}} G_{k+1} = -\xi_i e_i^{\mathrm{T}} + e_i^{\mathrm{T}} \left(Q \circ (U(0) - I) + \int_0^\infty (Q \circ \mu(x)) e^{G_k x} dx \right) L_i^*(G_k),$$

for $i = 1, \ldots, n$ and $k = 0, 1, \ldots$, where e_i is the i-th column of the identity matrix, $\xi_i = |s_i|$, $L_i^*(V) = (\xi_i I + V)(|q_{ii}| I - Y_i(V))^{-1}$ and $Y_i(V) = a_i V + \frac{1}{2}\sigma_i^2 V^2 + \int_0^\infty (e^{Vx} - I)\nu_i(x)dx$.

For the numerical experiments we considered two problems that we describe below.

Example 1. The phase generator is the $n \times n$ matrix

$$Q = \begin{bmatrix} -\alpha & \alpha & & \\ & \ddots & \ddots & \\ & & \ddots & \alpha \\ \alpha & & & -\alpha \end{bmatrix}.$$

The Brownian motion is defined by $a = \rho\mathbf{1}$, with $\rho < 0$, and $\sigma = \mathbf{1}$. There is no jump when there is a change of phase, hence $U(0) = \mathbf{1}^{\mathrm{T}} \cdot \mathbf{1}$ and $\mu(x) = 0$ for all $x > 0$. Jumps during phase i, for $1 \leq i \leq n$, all have the same density $\nu(x) = \lambda\boldsymbol{\tau}^{\mathrm{T}} e^{Tx}(-T\mathbf{1})$ where $(\boldsymbol{\tau}, T)$ is a phase-type distribution used in [6]: define the $\ell \times \ell$ transition matrix

$$\widehat{T} = \begin{bmatrix} -(c+s) & (1/r_1) & (1/r_1)^2 & \cdots & (1/r_1)^{\ell-1} \\ (r_2/r_1) & -(r_2/r_1) & & & \\ (r_2/r_1)^2 & & -(r_2/r_1)^2 & & \\ \vdots & & & \ddots & \\ (r_2/r_1)^{\ell-1} & & & & -(r_2/r_1)^{\ell-1} \end{bmatrix},$$

where $s = (1/r_1) + (1/r_1)^2 + \cdots + (1/r_1)^{\ell-1}$, and the empty entries are zeros. The parameters must satisfy the conditions $r_1 > 1$, $r_1 > r_2 > 0$, $c > 0$. The initial probability vector is $\boldsymbol{\tau}^{\mathrm{T}} = [1, 0, \cdots, 0]$ and the matrix T is defined as

$$T = (-\boldsymbol{\tau}^{\mathrm{T}}\widehat{T}^{-1}\mathbf{1})\widehat{T},$$

such that the expected jump size is $-\boldsymbol{\tau}^{\mathrm{T}} T^{-1} \mathbf{1} = 1$. The overall drift is $\kappa = \rho + \lambda$.

We take the parameters $r_1 = 2$, $r_2 = 1$, $c = 1.5$ and $\ell = 10$ like in [6], and $\lambda = 0.1$, $\alpha = 1$, $n = 8$, $\rho = -1$, so that the Brownian drift is constant and equal to -1. With these parameters the value of τ^* in (4.9) is $\tau^* \approx 1.27$ by using the quadratic equation in (4.7), it is $\tau^* \approx 1.33$ by using (4.8).

We have compared the convergence speed of the U-based, QME-based and NARE algorithms with $X_0 = 0$ and, for the U-based algorithm, with different values of τ. For the QME-based and NARE-based algorithm we have chosen the value $\tau_{opt} = 1.33$. In Figure 7.1 we report the difference $\delta_k = \|X_k - X_{k-1}\|_\infty$ as a function of k, where X_k is the approximation at step k provided by each algorithm.

We observe that the QME and NARE-based algorithms have the same convergence rate and are fastest. The convergence of the U-based algorithm depends on τ and reaches the fastest speed with the largest value of τ. The actual convergence rates of the QME and NARE-based algorithms, and U-based algorithm with $\tau = \tau_{opt}$, coincide with the corresponding theoretical estimates, that is $\rho(R)$ and $\rho(\widehat{R})$ of (6.7)
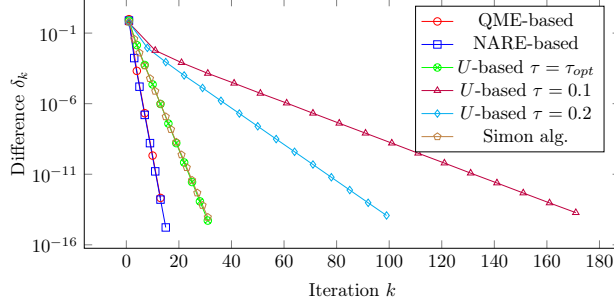
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FIG. 7.1. *Example 1. Difference $\delta_k = \|X_k - X_{k-1}\|_\infty$ per step for different algorithms, with $X_0 = 0$. In this example $\tau_{opt} = 1.33$.*

and (6.12), respectively, namely 0.10 for the QME and NARE-based algorithms, 0.35 for the U-based algorithm.

By choosing as starting approximation $X_0 = I$ for the QME and U-based algorithms, and $X_0 = \Delta_b$ for the NARE-algorithm, the convergence is faster, as shown in Figure 7.2. As in the case $X_0 = 0$, the QME and NARE-algorithms have the same convergence rate and faster convergence than the U-based algorithm.
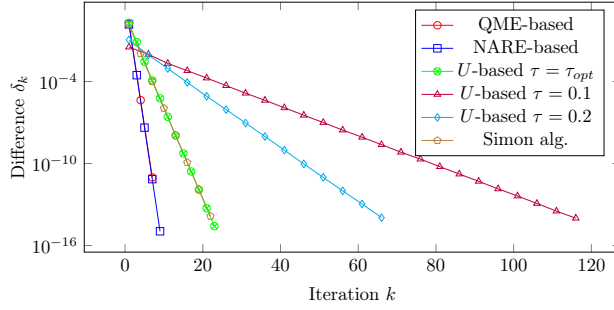

FIG. 7.2. *Example 1. Difference $\delta_k = \|X_k - X_{k-1}\|_\infty$ per step for different algorithms, with $X_0 = I$ for the QME-based algorithms, $X_0 = \Delta_b$ for the NARE-based algorithm. In this example $\tau_{opt} = 1.33$.*

The faster convergence of the sequence generated by setting $X_0 = I$, or $X_0 = \Delta_b$, is not a surprise in fact the same behaviour is encountered in the standard fixed point iterations for power series matrix equations. We have not performed a theoretical convergence analysis in this case. In our opinion it should be possible to carry out this analysis by following the same arguments and tools as in [5].

In Table 7.1 we report the CPU time required by the different algorithms, with $X_0 = 0$, and $X_0 = I$ or $X_0 = \Delta_b$ (for the NARE-based and the Simon algorithms). We observe that QME and NARE algorithms perform similarly, and are substantially faster than the U-based algorithm. A further acceleration is obtained by choosing $X_0 = I$ and $X_0 = \Delta_b$ for the QME and NARE-based algorithm, respectively. In Figures 7.1 and 7.2, and in Table 7.1, we systematically see that the Simon algorithm and the best U-based algorithm have similar performances. The Breuer algorithm is significantly slower for $X_0 = 0$.

In Table 7.2 we compare the residual error $\|F(\tilde{X})\|_\infty$ for the approximation $\tilde{X}$ obtained by using the QME-based algorithm with different values of τ. We observe

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X_0	U-based	QME	NARE	Simon alg.	Breuer alg.
0	1.6	0.77	0.74	1.55	12.0 (0.46)
I/Δ_b	1.15	**0.42**	**0.42**	1.15	–

<div align="center">TABLE 7.1</div>

Example 1. CPU time (in seconds) for the different algorithms, with starting approximation $X_0 = 0$ and $X_0 = I$ for U-based and QME-based, $X_0 = \Delta_b$ for the NARE-based and Simon algorithm. For the Breuer algorithm the preprocessing time is given in parenthesis.

that smaller values of τ yield lower accuracy, therefore the value τ_{opt} is the best choice.

	τ_{opt}	$\tau = 10^{-1}$	$\tau = 10^{-3}$	$\tau = 10^{-5}$	$\tau = 10^{-7}$
Residual	5.9e-16	1.0e-14	7.9e-11	1.0e-7	1.0e-2

<div align="center">TABLE 7.2</div>

Example 1. Residual error for different values of τ for the QME-based algorithm with $X_0 = 0$.

Finally, we have compared the QME-based algorithm to the Simon algorithm on the basis of CPU time and residual error in the case of Example 1 for increasing values of the matrix size n. As it turns out from Table 7.3, the QME-based algorithm is faster than Simon algorithm by a factor between 2.5 and 3, and provides a residual error which is smaller of one order of magnitude with respect to Simon algorithm.

Example 2. This problem is inspired from [12]. The level normally evolves in Phases 1 or 2; occasionally it is in Phase 3 for short periods of time. In Phases 1 and 2, the volatility is moderate, the drift is positive in Phase 1, negative in Phase 2. There is an exponential jump when the phase moves from 1 to 2 and back. In Phase 3, the volatility is high, the drift is negative, and there are repeated jumps. The infinitesimal generator is

$$Q = \begin{bmatrix} -\alpha - \omega & \alpha & \omega \\ \alpha & -\alpha - \omega & \omega \\ \beta & \beta & -2\beta \end{bmatrix}$$

with $\alpha = 1$, $\omega = 0.25$, $\beta = 0.5$, so that the process spends 1/5th of the time in Phase 3. Moreover, $a^{\mathrm{T}} = [-2, 1, -\gamma]$, $\sigma^{\mathrm{T}} = [1, 1, 10]$, where $\gamma > 0$ is a parameter that will allow us to change the dynamics of the process, and

$$\mu(x) = \begin{bmatrix} \cdot & \eta e^{-\eta x} & \cdot \\ \eta e^{-\eta x} & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$

where $\eta > 2$, $\nu_1(x) = \nu_2(x) = 0$ for all $x \geq 0$, and $\nu_3(x) = \gamma e^{-x}$. The asymptotic drift is

$$\kappa = \beta(2\alpha/\eta - 1)/(\omega + 2\beta).$$

We have chosen $\gamma = 10^{-4}$ and $\eta = 4$.

The observations, in terms convergence, CPU time and accuracy, are the same as for Example 1. We report in Figure 7.3 the error as a function of the number of iterations of the QME, the U-based algorithms with $X_0 = 0$ and $X_0 = I$, and the Simon algorithm with $X_0 = 0$ and $X_0 = \Delta_b$. The fastest convergence is obtained by the QME-based algorithm with $X_0 = I$ or $X_0 = \Delta_b$.

For this problem also, the actual convergence rates of the QME and U-based algorithm with $\tau = \tau_{opt}$, coincide with the corresponding theoretical estimates, that

	n	10	20	40	80	160	320	640
	QME	0.46	0.75	2.12	4.95	18.1	69.7	409.5
cpu	Simon	1.28	2.04	6.28	13.6	46.3	176.5	1023.5
	ratio	2.8	2.7	3.0	2.7	2.6	2.5	2.5
error	QME	7.0e-16	7.0e-16	1.0e-15	1.1e-15	2.4e-15	6.5e-15	7.6e-15
	Simon	3.8e-15	8.3e-15	1.4e-14	3.0e-14	5.7e-14	1.1e-13	2.6e-13

TABLE 7.3

CPU time in seconds, and residual error, for the QME-based algorithm and the Simon algorithm concerning Example 1 for different values of the matrix size n.
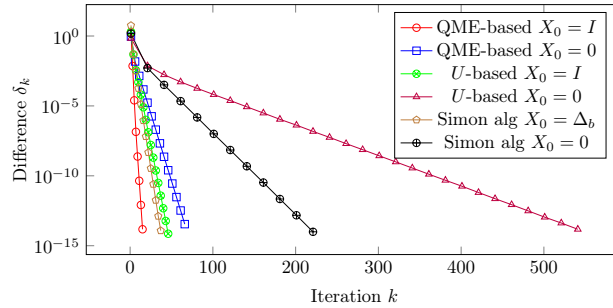

FIG. 7.3. *Example 2. Difference $\delta_k = \|X_k - X_{k-1}\|_\infty$ per step for different algorithms, with $X_0 = I$ for the QME-based algorithms, $X_0 = \Delta_b$ for the NARE-based algorithm. In this example $t_{opt} = 0.35$.*

is $\rho(R)$ and $\rho(\widehat{R})$ of (6.7) and (6.12), respectively, namely 0.62 for the QME-based algorithm, 0.95 for the U-based algorithm.

In Table 7.4 we report the CPU time required by the different algorithms, with $X_0 = 0$, $X_0 = I$ and $X_0 = \Delta_b$ for the NARE-based algorithm and Simon algorithm. Also in this case QME and NARE algorithms perform similarly, and are substantially faster than the U–based algorithm. A further acceleration is obtained by choosing $X_0 = I$ and $X_0 = \Delta_b$ for the QME and NARE-based algorithm, respectively. Compared with the Simon [30] and with the Breuer algorithm [7], the QME and NARE-based algorithms are faster.

X_0	U-based	QME	NARE	Simon alg.	Breuer alg.
0	11.6	1.4	1.5	4.4	1.7 (0.03)
I/Δ_b	0.96	**0.33**	**0.33**	0.78	–

TABLE 7.4

Example 2. CPU time (in seconds) for the different algorithms, with starting approximation $X_0 = 0$ and $X_0 = I / \Delta_b$. For the Breuer algorithm the preprocessing time is between parenthesis.

8. Conclusions. We have introduced some numerical methods for the solution of an integral matrix equation arising in Markov modulated Lévy processes. These methods rely on the solution of a sequence of quadratic matrix equations, which can be either unilateral or algebraic Riccati equations. We have performed a formal convergence analysis and shown the effectiveness of the proposed methods by an extensive numerical experimentation and by comparisons with some known iterative heuristics [7, 30].

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