Church Rosser λ-theories, infinite λ-terms and consistency problems *

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Abstract

We treat a general technique to obtain Church - Rosser extensions of the $\lambda\beta$-calculus, based on the notion of "confining class" and on an infinitary version of λ-calculus. We apply the technique to find a large class of terms which can be consistently equated to every other term, and we also show that many equations between λ-terms can be consistently added to the the $\lambda\beta$-calculus.

1 Introduction

We treat a general technique developed by the authors to obtain Church - Rosser consistent extensions of the $\lambda\beta$-calculus. (We do not consider here the $\lambda\beta\eta$-calculus.) Let M and N be two non-convertible closed λ-terms and assume that we want to prove that the theory $\lambda\beta + \{M = N\}$ is consistent, i.e. $\lambda\beta + \{M = N\} \vdash \mathbf{K} = \mathbf{S}$ does not hold (written $Con(M = N)$). An idea due to Mitschke (see [1] Section 15.3, [13]) is to obtain such consistency result *via* a stronger (but often easier to prove) statement: that some suitable theory $\lambda\beta\mu$, where μ is a notion of reduction, is a Church - Rosser extension of the theory $\lambda\beta$. The theory $\lambda\beta\mu$ has to enjoy the following properties: M is convertible to N (with

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respect to the extended notion of reduction), though **K** is still not convertible to **S**. Mitschke applied this idea to give a new proof of a result of Jacopini, which states that for every closed term M we have $Con(M = \mathbf{\Omega})$, i.e. $\mathbf{\Omega}$ is an *easy term.* This was obtained by adding (for a fixed M) the new reduction rule: $\mathbf{\Omega} \to M$ to the β-reduction rules.

In [7] it is shown how to extend such technique to terms much more complex than $\mathbf{\Omega}$ by "collapsing" a whole class of zero terms. As a corollary it was shown that the class of easy terms is not stable in the sense that if we equate all the easy terms to $\mathbf{\Omega}$, then some non-easy term is forced to be equal to an easy term.

The method is improved in [3], where the idea of using Böhm trees is also introduced. In this way many counterintuitive consistency results were obtained as a consequence of a general separation theorem: given a solvable closed term A, there is a closed zero term X such that $\neg Con(X = A)$ and for every closed term B whose infinite β-Böhm tree is incompatible with the one of A, $Con(X = B)$ holds. As a corollary we have: (1) there is a closed term X which can be consistently equated to every closed term $A \neq_\beta \lambda x.x$ but not to $\lambda x.x$; (2) there is a closed term that can be consistently equated to every closed normal form, but not to every closed term.

Here we take further this semantical approach using, instead of Böhm trees, the infinitary λ-calculus developed in [2]. Our goal is to obtain Church-Rosser extensions of finite λ-calculus, and we use infinite λ-calculus only as a tool to obtain such extensions. We believe that the consistency results we obtain in section 5 would be very difficult to prove without this detour into infinitary λ-calculus.

To keep the paper self-contained we recall the main definitions and results of infinite λ-calculus in section 3. We also include a new characterization of mute terms, namely those terms whose infinite normal form is $\perp$.

Infinite λ-calculus was independently introduced by Klop, Kennaway and F-J. de Vries in a recent preprint [12]. Our approach is different since we do not equate all the unsolvable terms.

This paper is meant to be a continuation of the line of research developed by the two authors in the papers quoted in the bibliography, and we refer to those papers for further motivations and results.

Notations: The sign $\equiv$ between terms stands for syntactical identity up to renaming of bound variables (α-conversion). The arrow $\to$ denotes one-step β-reduction, $\to^*$ is its transitive closure, and $=$ stands for provable equality in some theory (in most of the cases $=$ is β-convertibility). For other notions of reduction extending β-reduction we use subscripts like, $\to_\rho$. We write $\to_{=\rho}$ for the symmetric closure of $\to_\rho$ so that $A \to_{=\rho} B$ if and only if $A \to_\rho B$ or $A \equiv B$. We write $\to_{\sigma\rho}$ for the notion of reduction generated (taking closure under substitutions and contexts) by the union of $\to_\sigma$ and $\to_\rho$.

A *context* $C[\ ]$ is a term containing some occurrences of a special constant $\square$ called "hole". $C[B]$ is the term obtained by replacing all the occurrences of the holes with the term B. With the notation $A[x := B]$ we indicate as usual the result of substituting all the free occurrences of x in A with B after renaming the bound variables of A in such a way that the free variables of B do not become bound in $A[x := B]$. The difference between substitutions and contexts is that the free variables of B might become bound in $C[B]$ but not in $A[x := B]$.

These notions make sense also for the infinite λ-terms introduced in section 3, in which case $C[\,]$ is allowed to contain any number of holes, possibly infinitely many.

A *trivial context* $C[\,]$ is a context consisting only of the hole: $C[\,] \equiv \Box$. A *collapsing context* is a context β-reducible to the hole, or equivalently $C[x] \rightarrow^* x$ where x is a fresh variable.

We write $Con(X = M)$ to mean that $\lambda\beta + \{X = M\}$ is a consistent λ-theory. An *easy* term is a term X such that for every closed term M, $Con(X = M)$ holds.

We make use of the following λ-terms: $\mathbf{K} \equiv \lambda xy.x$, $\mathbf{S} \equiv \lambda xyz.xz(yz)$, $\mathbf{\Omega} \equiv (\lambda x.xx)(\lambda x.xx)$, $\mathbf{\Omega}_3 \equiv (\lambda x.xxx)(\lambda x.xxx)$, $\mathbf{Y} \equiv (\lambda hx.x(hhx))(\lambda hx.x(hhx))$.

For technical reasons we expand the language of λ-calculus with a special constant $\perp$.

2 Top normal forms and mute terms

In this section we recall the notion of top normal form of a lambda term defined in [2] and the associated notion of top reduction. Since top reductions are special cases of head reductions, there will more terms with a top normal form (i.e. normal form with respect to top reduction) than terms with an head normal form. To avoid complications we start by working in the usual (i.e. finite) $\lambda\beta$-calculus, namely we do not consider for the moment either infinite terms or infinite reductions. We recall that a *zero term* is a term which cannot be β-reduced to an abstraction term, i.e. to a term of the form $\lambda x.A$. Examples of such terms are $\mathbf{\Omega} \equiv \omega\omega \equiv (\lambda x.xx)(\lambda x.xx)$, $\mathbf{\Omega}_3 \equiv \omega_3\omega_3 \equiv (\lambda x.xxx)(\lambda x.xxx)$ and the special constant $\perp$. A *top normal form* (t.n.f) is a term of one of the following three kinds:

1. a variable;
2. an abstraction term $\lambda x.B$;
3. an application term of the form AB where A is a zero term.

It is easy to see that any β-reduct of a t.n.f is again a t.n.f., and moreover it is a t.n.f. of the same kind. To see this it suffices to observe that if A is a zero term, then any β-reduct of AB has the form $A'B'$ with $A \rightarrow^* A'$ and $B \rightarrow^* B'$. This justifies the name top normal form. We say that a term *has a top normal form* if it can be reduced to a term in top normal form.

It follows from the Church-Rosser property that if a term A has a top normal form of the i-th kind ($i = 1, 2, 3$), then all its top normal forms are of the i-th kind. It is not possible for instance that a term can be reduced both to a term of the form $\lambda x.B$ and to a term of the form UV with U a zero term.

There are terms, for instance $\mathbf{\Omega}$, which do not have a top normal form. Such terms are called *mute*. Notice that the special constant $\perp$ is a mute term.

Not all zero terms are mute: the β-reduction $\mathbf{\Omega}_3 \to \mathbf{\Omega}_3\omega_3$ shows that the zero term $\mathbf{\Omega}_3$ has a top normal form of the applicative kind, namely $\mathbf{\Omega}_3\omega_3$.

A defect of the notion of top normal form, if compared with the notion of head normal form, is that it is not constructive: there is no algorithm to test whether a term is a zero term, hence whether an applicative term *is* in top normal form (it is a co-recursively-enumerable predicate). To *have* a top normal form, is even less constructive (it is a $\exists\forall$ predicate in the arithmetical hierarchy). This should not deter us from working with top normal forms since in many cases, like the case of $\mathbf{\Omega}_3$, they can be easily computed.

Associated with top normal forms is the notion of top reduction. We say that a β-reduction $A \to B$ is a *top reduction* if the redex $\Delta \subset A$ being contracted is A itself. Top reductions are special cases of head reductions (see [2]), i.e. reductions of the form

$$\lambda x_1, \ldots x_n.(\lambda y.U)VM_1 \ldots M_k \to \lambda x_1 \ldots x_n.U[x := V]M_1 \ldots M_k.$$

Indeed top reductions are the special case $n = k = 0$.

It follows from the definitions that a term A is in top normal form if and only if it is different from $\perp$ and there are no top reductions from A to some other term. The following result illustrates the relationship between mute terms and top reductions.

Theorem 2.1 *For a term A not β-convertible to $\perp$ the following are equivalent:*

1. *A is mute;*
2. *There is an infinite sequence of β-reductions $A \to A_1 \to A_2 \to \ldots$ which contains infinitely many top reductions (not necessarily consecutive). Moreover we can take such an infinite sequence consisting entirely of head-reductions.*

Before giving the proof we recall that if $s: A \to B$ and $r: A \to C$ are two β-reductions starting with the same term, then there is a well known way of defining a term D and two β-reductions $s/r: C \to D$ and $r/s: B \to D$ (see [1]). The reduction s/r is called the *projection* of s over r.

Proof of Theorem 2.1. We need a result from [2]: if a term has a top normal form, then it can be reduced to a top normal form by a finite sequence of head reductions. Granted this, suppose that A is not mute. Then A has a top normal form B which can be reached by a sequence of head reductions $A \to B_1 \to B_2 \to \ldots \to B_n \equiv B$. By taking projections it follows easily that *any* sequence of reductions starting from A and containing more than n head reductions (not necessarily consecutive), ends up in a reduct of B, hence still in a top normal form. This proves that 2 implies 1.

For the converse it suffices to observe that if A is a mute term not β-reducible to $\perp$, then A must be an applicative term of the form UV with U not a zero term. It then follows that U can be reduced to an abstraction term $\lambda x.T$, hence A can be reduced to $(\lambda x.T)V$. We can now reduce this last term to $T[x := V]$ via a top reduction. Repeating the argument starting from the mute term $T[x := V]$ we obtain an infinite sequence with the desired properties. QED.

3 Infinite λ-calculus

In this expository section we recall the definitions and results of [2] that we need in the sequel.

3.1 Infinite λ-terms

Our next goal is to introduce the "infinite normal form" of a (finite) λ-term. We will see that such a notion is similar to the infinite Böhm tree (see [1]), except that is is computed using top normal forms instead of head normal forms.

We identify λ-terms with their parsing trees, so we write them either in linear form or in tree form. An *infinite λ-term* is defined as a finite or infinite rooted tree with binary application nodes "@", unary abstraction nodes "λx" (where x is any variable), and leaves labeled either by a variable or by the constant "$\perp$". Each node of an infinite λ-term determines a subterm, namely the subterm whose root (in the tree-representation) is the given node. The infinite λ-terms include as special cases the finite ones.

Infinite λ-terms arise in a natural way as "limits" of infinite sequences of β-reductions if we try to compute top normal forms "hereditarily". So if we start with $\mathbf{\Omega}_3$ we obtain the top normal form $\mathbf{\Omega}_3\omega_3$ and continuing in the same fashion we generate the infinite sequence of β-reductions $\mathbf{\Omega}_3 \to \mathbf{\Omega}_3\omega_3 \to (\mathbf{\Omega}_3\omega_3)\omega_3 \to$ etc. It is natural to take some kind of limit of this process and to set $\mathbf{\Omega}_3 \to_\infty ((((\ldots)\omega_3)\omega_3)\omega_3)\omega_3$ (infinitely many ω_3's). In tree-form:

$\mathbf{\Omega}_3 \to_\infty$

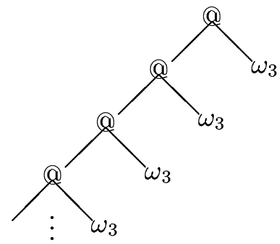

Figure 1

Notice that the infinite λ-term on the right is a normal form because it has no β-redexes (i.e. subterms of the form $(\lambda x.A)B$). It is natural to call it the infinite normal form of $\mathbf{\Omega}_3$. Thus we see that contrary to what happens in the usual (finite) λ-calculus, there are infinite λ-terms which are in normal form and yet they begin with an application node.

3.2 Finite β-reductions among infinite terms

β-normal forms are defined for infinite terms as for finite ones: namely as terms not containing subterms of the form $(\lambda x.A)B$.

β-reduction is also defined as for finite terms, namely as the process of replacing a subterm of the form $(\lambda x.A)B$ with $A[x := B]$ (with renaming of bound variables to avoid conflicts).

The Church-Rosser property fails in a trivial way for β-reductions among infinite terms. Indeed take $A \equiv x(x(x(\ldots)))$ (infinitely many x's) and let B be a term which can be β-reduced to, say, B'. Then $(\lambda x.A)B$ can be β-reduced to $B(B(B(\ldots)))$ (infinitely many B's). On the other hand $(\lambda x.A)B$ can also be reduced to $(\lambda x.A)B'$, hence to $B'(B'(B'(\ldots)))$. Now it is clear that we cannot find a common reduct of $B(B(B(\ldots)))$ and $B'(B'(B'(\ldots)))$ via a finite sequence of β-reductions, and the Church-Rosser property fails.

In section 3.3 we will introduce a notion of infinite β-reduction in such a way that $B(B(B(\ldots)))$ can be reduced to $B'(B'(B'(\ldots)))$ via an infinite β-reduction. This however *will not* solve all our problems with the Church-Rosser property (as a more complicated example will show) and a suitable rule concerning $\perp$ must be added to restore it.

To stress the difference between β-reduction and the infinite β-reduction of section 3.3 we sometimes call the former one *finite* β-reduction. So finite β-reduction is either one-step β-reduction $\to$ or its transitive closure $\to^*$.

The notions of zero term and top normal can be defined for infinite terms in exactly the same way as for finite ones, namely using *finite* β-reductions. So an infinite term A is a *zero term* if and only if there is no finite β-reduction from A to an abstraction term. An infinite term is a *top normal form* if and only if it is either a variable, or an abstraction term, or a term of the form BC with

B a zero term. An infinite term *has a top normal form* if it can be reduced to a top normal form by a finite β-reduction. An infinite term is *mute* if it has no top normal form.

3.3 Infinite β-reduction

We will define infinite β-reduction among infinite terms. So among infinite terms we have both a notion of finite β-reduction $\rightarrow^*$, and a notion of infinite β-reduction $\rightarrow_\infty$.

Definition 3.1 Given two infinite terms A and B, we say $A \equiv_n B$ if A and B coincide up to the n-th level of their tree-representation. More precisely:

1. $A \equiv_0 B$ iff A and B have the same root.

2. $A \equiv_{n+1} B$ iff A and B have the same root and each immediate subterm of A is in relation $\equiv_n$ to the corresponding immediate subterm of B.

We recall that the root of an infinite term is either an application node @, or an abstraction node λx, or a variable x, or the constant $\bot$ (in the last two cases the whole term coincides with its root). Note that $A \equiv B$ iff $A \equiv_n B$ for every n. We agree that for n negative $A \equiv_n B$ always holds. Since infinite terms are identified with trees, they possess a natural topology and a notion of limit.

Definition 3.2 Let $< A_n \mid n \in \omega >$ be a sequence of infinite terms. We say $\lim < A_n > \equiv A$ if A is an infinite term, and $\forall k \exists n \forall m \geq n \;\; A_m \equiv_k A$.

Definition 3.3 The *depth* of a specific occurrence of a subterm Δ in A, is defined as the length of the path connecting the root of Δ to the root of A in the tree-representation of A. The depth of a β-reduction $A \rightarrow B$ is defined as the depth of the redex $\Delta \subset A$ being contracted.

So for instance the depth of A in A is 0. The depth of A in $\lambda x.A$ is 1. The depth of A in $\lambda x.AB$ and in $\lambda x.BA$ is 2. The depth of A in $\lambda x.ABC$ is 3. Notice that if the reduction $A \rightarrow B$ has depth n, then $A \equiv_{n-1} B$.

Definition 3.4 Let $s\colon A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow \ldots$ be an infinite sequence of β-reductions. We say that s *converges* to the infinite term B if:

1. $B = \lim < A_i >$;

2. the depth of the reduction $A_i \rightarrow A_{i+1}$ tends to infinity with i.

So for an infinite sequence of reductions to converge we require not only that the terms in the sequence converge, but also that the depth of the reductions tends to infinity. This is called *strong convergence* in the context of term rewriting systems [11]. To illustrate the difference consider a term H, constructed by the help of a fixed point combinator, such that $Hx \to H(\mathbf{s}x)$ where $\mathbf{s}$ is a λ-representation of the successor function. Then $H\mathbf{0} \to H(\mathbf{s0}) \to H(\mathbf{s}(\mathbf{s0})) \to \dots$ etc. The terms in this reduction converge to the infinite term $H(\mathbf{s}(\mathbf{s}(\dots)))$ (infinitely many $\mathbf{s}$'s), but the sequence of reductions itself does not converge because the depth of the redexes does not tend to infinity.

Definition 3.5 We define $A \to_\infty B$ (*infinite β-reduction*) if and only if either $A \to^* B$ or there is an infinite sequence of reductions starting from A and converging to B.

In the previous section we have seen an example of an infinite β-reduction starting from the term $\mathbf{\Omega}_3$. This was an example of an infinite β-reduction starting from a finite term and ending up in a β-normal form. Clearly not all infinite reductions have this property. We need the following fact:

Proposition 3.6 *$\to_\infty$ is transitive, i.e. $\to_\infty$ coincides with its transitive closure $\to_\infty^*$.*

3.4 Infinite $\beta\perp$-reductions

Infinite β-reduction $\to_\infty$ is not Church-Rosser. Indeed let $\mathbf{I} \equiv \lambda x.x$, and $A \equiv (\lambda x.\mathbf{I}(xx))(\lambda x.\mathbf{I}(xx))$. Then we have both $A \to_\infty \mathbf{I}(\mathbf{I}(\mathbf{I}(\dots)))$ (infinitely many $\mathbf{I}$'s) and $A \to^* (\lambda x.xx)(\lambda x.xx)$, and no common infinite β-reduct exists. It turns out that the mute terms are the only ones responsible for the failure of the Church-Rosser property.

Definition 3.7 We define a *$\perp$-redex* as a mute term different from $\perp$. Define now $\to_\perp$ as the least notion of reduction which contains $\to$ (β-reduction) and sends all the mute terms to $\perp$ (it can be proved that every substitution instance of a mute term is mute, so we do not need to take closure under substitutions). Now define $\to_{\perp\infty}$ exactly as $\to_\infty$ but starting from $\to_\perp$ instead of $\to$.

It can be shown that $\to_{\perp\infty}$ is transitive. Notice that if a term has no β-redexes, it has no $\perp$-redexes as well. So the notion of β-normal form coincides with the notion of $\beta\perp$-normal form and we can speak simply of normal form.

Definition 3.8 We say that an infinite term B *arises from a finite term* if there is a finite term A with $A \to_{\perp\infty} B$.

So the infinite term $(((\dots)\omega_3)\omega_3)\omega_3$ arises from the finite term $\mathbf{\Omega}_3$.

Theorem 3.9 *([2])*

1. *$\to_{\perp\infty}$ is Church-Rosser if restricted to terms B which arise from a finite term.*

2. *Every infinite term A can be reduced to a β-normal form via $\to_{\perp\infty}$. This β-normal form is unique if A arises from a finite term.*

Definition 3.10 If A arises from a finite term, we denote by A^∞ the unique β-normal form such that $A \to_{\perp\infty} A^\infty$ and we call A^∞ the *infinite normal form* of A.

Thus $\mathbf{\Omega}_3^\infty \equiv (((\ldots)\omega_3)\omega_3)\omega_3$. We believe that Theorem 3.9 holds without the restriction to terms arising from finite terms. A possible way to prove this fact is to use transfinite reductions of length bigger than the ordinal number ω as in [11] [1].

The easy part of the theorem, namely the *existence* of infinite normal forms, holds without the restriction to terms arising from finite terms and follows easily from the following proposition.

Proposition 3.11 1. *If $A \to^* x$, then $A^\infty \equiv x$;*

2. *if $A \to^* \lambda x.B$, then $A^\infty \equiv \lambda x.B^\infty$;*

3. *if $A \to^* BC$ and B is a zero term, then $A^\infty \equiv B^\infty C^\infty$;*

4. *if A is mute (which amounts to say that 1,2,3 do not apply), then $A^\infty \equiv \perp$.*

The hypothesis that B is a zero term in clause 3 ensures that BC is a top normal form of A (hence all the β-reducts of BC are of the form $B'C'$ with $B \to^* B'$ and $C \to^* C'$). Notice that if A has a β-normal form B with respect to finite β-reduction, then $A^\infty \equiv B$. So infinite normal forms are a generalization of normal forms. We have $\perp^\infty \equiv \perp$, $(\lambda x.\ \perp)^\infty \equiv \lambda x.\ \perp$ and $(\perp\perp)^\infty \equiv \perp\perp$. This is to be contrasted with the theory of infinite Böhm trees where we have $\perp\perp \equiv \lambda x.\ \perp \equiv \perp$.

A basic difference between infinite normal forms and infinite Böhm trees is that an infinite normal form can begin with an application node (as in the case of $\perp\perp$ and the infinite normal form of $\mathbf{\Omega}_3$). Another difference is that an infinite normal form can begin with infinitely many λ's. This is the case of the infinite normal form of the term $\mathbf{YK}$, where $\mathbf{Y} \equiv (\lambda hx.x(hhx))(\lambda hx.x(hhx))$ is the Turing fixed point combinator and $\mathbf{K} \equiv \lambda xy.x$. Indeed we have $\mathbf{Y}^\infty \equiv \lambda x.x(x(x(\ldots)))$ (which is essentially the Böhm tree of $\mathbf{Y}$) and $\mathbf{YK} \to_\infty \lambda y_0(\lambda y_1(\lambda y_2(\ldots)))$ (infinitely many λ's).

Sometimes it is not so easy to compute infinite normal forms due to the fact that we might not be able to test algorithmically whether clause 3 in Proposition 3.11 can be applied (as the notion of zero term is not decidable). So in general the computation of A^∞ requires some ingenuity. As an exercise the reader can verify that $(\mathbf{YY})^\infty$ is the complete infinite binary tree (with application nodes @ only, and no leaves).

[1] Added in proofs: alternatively one can work with reductions of length at most ω and obtain the infinite Church-Rosser theorem via a kind of standardization theorem.

3.5 Some properties of infinite reductions

In this section all the terms are assumed to be infinite terms arising from finite terms (unless otherwise stated). Under this hypothesis the Church-Rosser property for $\rightarrow_{\perp\infty}$ holds and we have existence and uniqueness of infinite normal forms. Clearly β-convertible terms have the same infinite normal form. Actually we have more:

Fact 3.12 *if $A \rightarrow_{\perp\infty} B$, then $A^\infty \equiv B^\infty$.*

We also need in the sequel the following result:

Fact 3.13 *if $A \rightarrow_{\perp\infty} B$, then A is a zero term iff B is a zero term. Hence A is a zero term iff A^∞ is an applicative term or $\perp$.*

Recall that a zero term is defined in terms of *finite* β-reductions, so the above result does not follow immediately from the definitions. Indeed Fact 3.13 has some not entirely trivial consequences concerning finite λ-calculus:

Corollary 3.14 *1. If A and N are two finite terms such that $A \rightarrow AN$, then A is a zero term.*

2. It is impossible for a finite term A to satisfy: $A \rightarrow AN$ and $A \rightarrow AM$ where N and M are two distinct finite terms.

Proof. Part 2 follows from part 1 and the Church-Rosser theorem. To prove part 1 assume $A \rightarrow AN$. Then we have the infinite sequence of reductions $A \rightarrow AN \rightarrow (AN)N \rightarrow ((AN)N)N \rightarrow \ldots$ etc., which converges to the infinite term $(((\ldots)N)N)N$ (infinitely many N's). So $A^\infty \equiv (((\ldots)N)N)N$, which is a zero term because any finite β-reduction starting from it can only happen inside the N's, so it cannot change the whole term to an abstraction term. QED

The notion of context makes sense also for infinite terms. In general a context $C[\,]$ may contain infinitely many holes. The infinite normal form $C^\infty[\,]$ of $C[\,]$ can be defined by treating the hole as if it were a new free variable and setting $C^\infty[\,] \equiv (C[\,])^\infty$. Even if $C[\,]$ is a context with only one hole, its infinite normal form $C^\infty[\,]$ may contain any number of holes, possibly zero or infinitely many.

Example 3.15 Let H be a term satisfying $Hx \rightarrow Hx(Hx)$. Then the infinite normal form of Hx is the complete infinite binary tree with application nodes only and no leaves. The variable x has disappeared because it has been pushed to infinity. So the context $C[\,] \equiv H\Box$ has an infinite normal form with no holes.

The possible presence of infinitely many holes makes the following fact not entirely trivial:

Fact 3.16 *(Assume all terms mentioned arise from finite terms)*

1. *If $B \to_{\perp\infty} B'$, and $C[\,] \to_{\perp\infty} C'[\,]$, then $C[B] \to_{\perp\infty} C'[B']$;*

2. $(C[B])^\infty \equiv (C^\infty[B^\infty])^\infty$.

3. *If B^∞ is a zero term, then $C[B]^\infty \equiv C^\infty[B^\infty]$.*

Proof. Part 2 follows immediately from Part 1 and the Church-Rosser property of $\to_{\perp\infty}$ observing that the two terms $(C[B])^\infty$ and $C^\infty[B^\infty]$ are both $(\to_{\perp\infty})$-reducts of $C[B]$. Part 3 follows by observing that if B is a zero term, then $C^\infty[B^\infty]$ is a normal form. For the proof of part 1 see [2]. QED

4 Church-Rosser Extensions

Up to Definition 4.10, we shall be concerned only with finite terms, and "term" means "finite term". Let X and M be two closed terms. It is well known that there is no algorithm to test whether $Con(X = M)$ holds. However we will give sufficient conditions which work in many cases.

Let us first make some reductions: it is known (see [1]) that two unsolvable terms can always be consistently equated. So we can assume that at least one of X and M, say M, is solvable (i.e. it has an head normal form). Böhm's famous theorem states that two $\beta\eta$-normal forms cannot be consistently equated [4], and actually any two terms with incompatible infinite $\beta\eta$-Böhm tree, cannot be consistently equated (see [1]). So we can assume that X has an infinite $\beta\eta$-Böhm tree compatible with the one of M (in the sense that replacing the bottom with suitable terms we can get the same Böhm tree). In particular X could be unsolvable. To simplify things we assume that X is the simplest kind of unsolvable: namely a zero term. So our general assumptions are: X is a zero term and M is solvable. Under these hypothesis we want to decide whether $Con(X = M)$ holds.

Since a zero term does not act on its arguments, the non-expert reader may think that the answer is always positive. However, this is not the case. The reason is that a zero term may give information simply by expanding itself! As an example, $\mathbf{\Omega}_3$ β-reduces to $\mathbf{\Omega}_3\omega_3$. So, e.g. in the theory $\lambda\beta + \{\mathbf{\Omega}_3 = \mathbf{K}\}$ we obtain $\mathbf{K} = \mathbf{K}\omega_3$ and the theory turns out to be inconsistent. So we have to take into account all the β-reducts of X. In particular, some reducts may be in the form $C[Y]$, for some non trivial context $C[\,]$ and some Y β-convertible to X. Then, from the equality $X = M$, the equality $C[M] = M$ follows. Now, at an informal level, we have to distinguish between three different cases:

1. $C[\,]$ begins by a hole, i.e. it is of the form $\Box(N_1[\,])\dots(N_k[\,])$. In this case the theory $\lambda\beta + \{X = M\}$ cannot be consistent for every choice of M (to get an inconsistent theory choose a suitable projector); it can be however consistent for particular choices of M. (The above example falls in this case, since $\mathbf{\Omega}_3$ gives rise to the context $\Box\omega_3$.)

2. Any such context $C[\ ]$ has a prefix (i.e. it does not begin with an hole) but we are not able to detect if $M = C[M]$ is consistent.

3. Any such context $C[\ ]$ has a prefix and moreover the form of such prefix is such that we expect that no contradiction can arise from $M = C[M]$.

The following makes precise the intuition corresponding to Case 3. Later we discuss some examples belonging to Case 2.

Definition 4.1 (Confining Classes I)
Given a closed term M, a class $\mathcal{C}$ of closed terms is called a *confining class* for M if the following holds:

1. every element of $\mathcal{C}$ is a zero term;

2. $\mathcal{C}$ is closed under β-reduction, i.e. if $B \in \mathcal{C}$ and B β-reduces to B_1 then $B_1 \in \mathcal{C}$;

3. for every non-trivial context $C[\ ]$, if $C[B] \in \mathcal{C}$ and $B \in \mathcal{C}$, then $C[M] \in \mathcal{C}$.

Theorem 4.2 *Given terms M and X, if there exists a confining class $\mathcal{C}$ for M such that $X \in \mathcal{C}$ then $Con(M = X)$.*

According to our informal discussion the third clause corresponds to the case in which the prefix is such that $C[M]$ is still a zero term. This is enough to obtain consistency. In fact we obtain more:

Theorem 4.3 *([3] Theorem 3.4) Let M be a term and $\mathcal{C}$ a confining class for M. Let μ be the notion of reduction generated (taking substitutions and contexts) from the following reduction rules: for every $B \in \mathcal{C}, B \to_\mu M$. Then the notion of reduction $\beta\mu$ is Church-Rosser.*

Remark 4.4 It is easy to see that Theorem 4.2 is a corollary of Theorem 4.3. In fact if $X \in \mathcal{C}$, then X is obviously $\beta\mu$-convertible to M. On the other hand $\mathbf{K}$ and $\mathbf{S}$ are still in normal form since they contain neither β- nor μ-redexes (observe that a μ-redex must be a zero term). Thus $\mathbf{K}$ and $\mathbf{S}$ are not $\beta\mu$-convertible (as $\beta\mu$-reduction is Church-Rosser) and we obtain the desired consistency result. Theorem 4.3 actually gives more: it is consistent with the $\lambda\beta$-calculus to simultaneously equate all the terms in $\mathcal{C}$ to M.

For later reference we sketch the proof of Theorem 4.3 (see [3], Theorem 3.4, for details).

Proof of Theorem 4.3
Claim: $\to_{=\mu}$ *is Church-Rosser and therefore* $\to_\mu^*$ *is Church-Rosser.*

To prove the claim observe that the case of disjoint μ-redexes is obvious. So let T and $C[T]$ two nested μ-redexes. We have: $C[M]\ {}_\mu\!\leftarrow C[T] \to_\mu M$, but

$C[M] \in \mathcal{C}$ so the diagram can be completed by reducing $C[M]$ to M (plus an empty reduction).
Claim: *$\to_\beta^*$ and $\to_\mu^*$ commute.*

To prove the claim observe that:

Fact: if we have: $C \; {}_\mu\!\leftarrow A \to_\beta B$, then there is a term D and reductions $C \to_{=\beta} D \; {}^*_\mu\!\leftarrow B$ (β-reductions can create many copies of, leave untouched or erase μ-redexes, but cannot make a μ-redex a non-redex).

Now we can use the strip method: start with the diagram: $C_\mu\!\leftarrow A \to_\beta A_1 \to_\beta \ldots \to_\beta A_n$ and use the above fact to get a shorter $\to_\beta^*$ reduction; so (arguing by induction) the claim follows.

To conclude the proof, observe that since $\to_\beta^*$ is Church-Rosser, $\to_\beta^*$ and $\to_\mu^*$ are a pair of commuting Church-Rosser notions of reduction. Therefore by the Hindley-Rosen Lemma (Lemma 3.3.6 of [1]), $\to_{\beta\mu}^*$ is Church-Rosser. QED

For applications see [3]. It must be pointed out that it is not at all clear how to choose a class $\mathcal{C}$ which works for a given term M and a given zero term X. In particular, to apply Theorem 4.2, we need a class closed under β-reduction (but not necessarily under β-convertibility). One source of complications is that some β-reduct of M may contain X as a subterm, and so new potential redexes arise. We will prove a strengthening of Theorem 4.3, which will later allow us to use infinite normal forms in the definition of the confining class. Recall that $C[\,]$ is called a collapsing context if $C[x] \to^* x$, where x is a fresh variable. Clearly every trivial context is collapsing.

Definition 4.5 (Confining Classes II)
Given a closed term M, a class $\mathcal{C}$ of closed terms is called a *weak confining class* for M if the following holds:

1. every element of $\mathcal{C}$ is a zero term;
2. $\mathcal{C}$ is closed under β-reduction, i.e. if $B \in \mathcal{C}$ and B β-reduces to B_1, then $B_1 \in \mathcal{C}$;
3. for every non-collapsing context $C[\,]$, if $C[B] \in \mathcal{C}$ and $B \in \mathcal{C}$, then $C[M] \in \mathcal{C}$.

The intuitive meaning of Definition 4.5 (with respect to Definition 4.1) is that we are no more afraid of possible "circular" behaviour of X (say $X \to_\beta^* \mathbf{I}X$), which gives rise to collapsing but non trivial contexts (e.g. $\mathbf{I}\Box$). Notice that if $C[\,]$ is collapsing then for every closed N, N and $C[N]$ have the same infinite normal form. The following theorem states that a weak confining class is enough to get the required Church-Rosser extension.

Theorem 4.6 *Let M be a closed term and $\mathcal{C}$ a weak confining class for M. Moreover, let μ be the notion of reduction obtained from the following reduction* *rules: for every $B \in \mathcal{C}$, $B \to_\mu M$. Then the notion of reduction $\beta\mu$ is Church-Rosser.*

As usual we close $\to_\mu$ under substitutions and contexts, so if $B \to_\mu M$, then $C[B] \to_\mu C[M]$ for every context $C[\ ]$ (closure under substitutions is not actually needed since $\mathcal{C}$ is a class of closed terms and M is also closed). The proof of Theorem 4.6 is a modification of the proof that all mute terms are easy [2]. Since it is rather complex we split it in a series of lemmas.

Lemma 4.7 *Any diagram $C\ {}_\mu\!\!\leftarrow A \to_\mu B$ can be extended in one of the following forms (for some term D):*

1. $B \to^*_\beta D \equiv C$;
2. $B \equiv D\ {}^*_\beta\!\!\leftarrow C$;
3. $B \to_{=\mu} D\ {}_{=\mu}\!\!\leftarrow C$

The lemma follows easily by considering the relative positions of the two μ-redexes and the definition of weak confining class. The main point to notice is that, unlike what happens in Theorem 4.3, μ-reduction is not Church-Rosser. In fact, if $C[B] \in \mathcal{C}$ and $B \in \mathcal{C}$, where $C[\ ]$ is a collapsing context, then $C[M]\,{}_\mu\!\!\leftarrow C[B] \to_\mu M$, but at this point a common μ-reduct may very well be lacking (since $C[M]$ is not necessarily in $\mathcal{C}$). To cope with this problem, we suitably extend μ to a larger notion of reduction ρ which turns out to be Church-Rosser.

Definition 4.8 **(ρ-reductions and collapsing reductions)**

1. Let c be the notion of reduction obtained from the following reduction rule: for every collapsing context $C[\ ]$, and for every closed term N, $C[N] \to_c N$.
2. Define $A \to_\rho B$ iff either $A \to_c B$ or $A \to_\mu B$.

We call c-reductions *collapsing reductions*. Notice that such a reduction can be simulated by a sequence of β-reductions.

Collapsing reductions were used in [2]. The proof of Theorem 4.6 is obtained through the following sequence of results which simplifies the treatment in [2] via the key observation that collapsing reductions are Church-Rosser. The proof of this fact is given in the appendix.

Lemma 4.9

1. *$\to_{=c}$ is Church-Rosser.*
2. *$\to_{=\rho}$ is Church-Rosser, and therefore $\to^*_{=\rho}$ is Church-Rosser.*

3. *Any diagram $C\ {}_{\mu}\!\!\leftarrow A \rightarrow_c B$ can be extended to one of the form $C \rightarrow_{=c} D\ {}_{\mu}\!\!\leftarrow B$ for some term D.*

4. *Any diagram $C\ {}_{c}\!\!\leftarrow A \rightarrow_\beta B$ can be extended to one of the form $C \rightarrow_{=\beta} D\ {}^{*}_{c}\!\!\leftarrow B$ for some term D.*

5. *Any diagram $C\ {}_{\mu}\!\!\leftarrow A \rightarrow_\beta B$ can be extended to one of the form $C \rightarrow_{=\beta} D\ {}^{*}_{\mu}\!\!\leftarrow B$ for some term D.*

6. *$\rightarrow^*_\beta$ and $\rightarrow^*_\rho$ commute.*

7. *$\rightarrow^*_{\beta\rho}$ is Church-Rosser.*

Proof. See the appendix. QED

Proof of Theorem 4.6. Since c-reductions can be simulated by sequences of β-reductions, $\rightarrow^*_{\beta\rho}$ coincides with $\rightarrow^*_{\beta\mu}$, hence it is Church-Rosser by the previous lemma. QED

As we observed above, the main problem is to find suitable confining classes. The following shows how this can be done starting from sematical considerations based on normal forms of infinite terms.

Definition 4.10 (Confining Classes III)
Given a closed finite term M, a class $\mathcal{C}_\infty$ of infinite closed terms arising from finite terms is called a *confining class of infinite terms* for M if the following holds:

1. every element of $\mathcal{C}_\infty$ is a zero term;

2. for every non-trivial infinite context $C^\infty[\ ]$ (possibly with infinitely many holes), if $C^\infty[B^\infty] \in \mathcal{C}_\infty$ and $B^\infty \in \mathcal{C}_\infty$, then $(C^\infty[M^\infty])^\infty \in \mathcal{C}_\infty$ provided that $C^\infty[M^\infty]$ arises from a finite term.

The restriction to terms arising from finite terms is only needed to ensure that the Church-Rosser property for $\rightarrow_{\perp\infty}$ reductions holds, and probably it can be omitted. The following result shows that when we come back to finite terms in the natural way, we get a weak confining class.

Theorem 4.11 *If $\mathcal{C}_\infty$ is a confining class of infinite terms for M and $\mathcal{C} = \{N \mid N$ is a finite closed term and $N^\infty \in \mathcal{C}_\infty\}$, then $\mathcal{C}$ is a weak confining class for M. (Hence it is consitent to simultaneously equate all terms in $\mathcal{C}$ to M.)*

Proof. We have to check clauses (1) - (3) of Definition 4.5. Observe that 4.5(1) follows from the fact that terms whose infinite normal form is a zero term are themselves zero terms (Fact 3.13). Moreover, from the fact that β-convertible terms have the same infinite normal form it follows that $\mathcal{C}$ is closed under β-reduction. Actually $\mathcal{C}$ is even closed under β-conversion (this was not necessarily so for the confining classes of Definition 4.1, namely those used in [3]).

Now, let $C[B] \in \mathcal{C}$, $B \in \mathcal{C}$ and let $C[\ ]$ be a non-collapsing context. This is the same as saying that the infinite normal form $C^\infty[\ ]$ of $C[\ ]$ is a non-trivial context (since $C[x]^\infty \equiv x$ iff $C[x] \to^* x$, where x is a fresh variable). We have to show that $C[M]^\infty \in \mathcal{C}_\infty$, i.e. $(C^\infty[M^\infty])^\infty \in \mathcal{C}_\infty$. Since B and $C[B]$ belong to $\mathcal{C}$, by definition of $\mathcal{C}$ we have $(C[B])^\infty \in \mathcal{C}_\infty$ and $B^\infty \in \mathcal{C}_\infty$. Moreover $(C[B])^\infty = (C^\infty[B^\infty])^\infty = C^\infty[B^\infty]$ by Fact 3.16. Now clause 2 in the definition of confining class of infinite terms gives us $C[M]^\infty \in \mathcal{C}_\infty$. QED

5 Applications

5.1 Self-similar Terms

In [2] a mute term is also called *zero term of degree zero*. One then defines a *zero term of degree* $n+1$ as a zero term β-convertible to a zero term of degree n applied to some other term. A zero term is said to be of *infinite degree* in the remaining cases. So $\mathbf{\Omega}$ is of degree zero, $\mathbf{\Omega K}$ is of degree one, and $\mathbf{\Omega}_3$ is of infinite degree. Since mute terms are easy and an easy term applied to any other term is easy, all zero terms X of finite degree are easy. So if one consider the question of whether a zero term X is easy, the only interesting case left is when X is a zero term of infinite degree. In this case X may or may not be easy. For instance $\mathbf{\Omega}_3$ is not easy (as we have seen that it cannot be equated to $\mathbf{K}$). On the other hand, as a first application of Theorem 4.11, we prove that $\mathbf{\Omega}_3\mathbf{I}$ (which is a another zero-term of infinite degree) is easy. More complex examples of such terms can be found in [5, 3].

Proposition 5.1 $\mathbf{\Omega}_3\mathbf{I}$ *is easy.*

Proof. Let $M \in \Lambda_0$ be given. As a confining class of infinite terms for M containing $\mathbf{\Omega}_3\mathbf{I}$ we simply take the class containig only $(\mathbf{\Omega}_3\mathbf{I})^\infty$, i.e. $\mathcal{C}_\infty = \{(\mathbf{\Omega}_3\mathbf{I})^\infty\}$. Clearly, $\mathcal{C}_\infty$ contains only zero terms. We have to show that if $C[B^\infty] \in \mathcal{C}_\infty$ and $B^\infty \in \mathcal{C}_\infty$, then $(C[M])^\infty \in \mathcal{C}_\infty$ or $(C[\ ])^\infty$ is the trivial context. The key point to observe is that the infinite term $(\mathbf{\Omega}_3\mathbf{I})^\infty$ has not any proper subterm identical to itself. To see this, just draw the picture:

$(\mathbf{\Omega}_3\mathbf{I})^\infty \equiv$

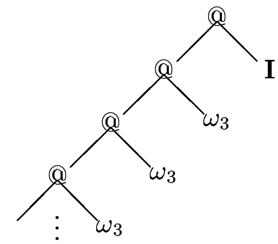

Figure 2

(By contrast, $\mathbf{\Omega}_3^\infty$ has infinitely many subterms identical to itself.) This shows that if $(C[B])^\infty \in \mathcal{C}_\infty$ and $B^\infty \in \mathcal{C}_\infty$, then $C[\;]^\infty$ has no holes or it is trivial. In the former case, $(C^\infty[M^\infty])^\infty \equiv (C^\infty[\,])^\infty \equiv C^\infty[\,] \equiv C^\infty[B^\infty] \in \mathcal{C}_\infty$. In the latter $(C[\,])^\infty$ is trivial and we are done. QED

The given proof of easiness is extremely simple, if compared with proofs in [7] or [3]. Of course, this depends on the machinery set up in the previous section. But we stress that, once this has be done, we can prove consistency or easiness results simply by looking at the infinite normal form of the terms involved. As an example, we can immediately generalize the above result to a large class of zero terms:

Definition 5.2 A (finite) closed term A is *self-similar* if A^∞ contains a proper subterm identical to itself.

For instance, $\mathbf{\Omega}_3$, $\mathbf{Y\Omega}_3$, $\mathbf{YY}$ are all self-similar, whilst $\mathbf{\Omega}_3\mathbf{I}$, $\mathbf{\Omega}_3\mathbf{K}$, $\mathbf{Y\Omega}_3\mathbf{I}$ are not self-similar (the "**I**" or "**K**" break the symmetry). In [3], a quite complex proof of the easiness of $\mathbf{Y\Omega}_3\mathbf{I}$ was given (in fact, the easiness of $\mathbf{Y\Omega}_3N$ was proved, with N a term with head normal form). Now, such results can be obtained from the following quite general proposition.

Theorem 5.3 *If $A \in \Lambda_0$ is a closed zero term which is not self-similar, then A is easy.*

Proof. Exactly as the proof of Proposition 5.1. Indeed in that proof we have only used the fact that $\mathbf{\Omega}_3\mathbf{I}$ is not self-similar. QED

As a corollary we obtain that every closed mute term is easy (as A is mute iff $A^\infty \equiv \perp$, and clearly $\perp$ has no proper subterms identical to itself). Actually the proof shows more: it is consistent to simultaneously equate all the closed mute terms to a fixed closed term M arbitrarily chosen (for a detailed proof of this fact see [2], where the result is proved even for non-closed mute terms).

We feel that theorem 5.3 is a nice formal counterpart of the general intuition that a zero term can give information about itself only by expanding itself (see the discussion at the beginning of the previous section). If a zero term is not self-similar, then also this way of communicating is not practicable, so we have no information at all and the term turns out to be easy.

5.2 A Consistency Result

The second application of Theorem 4.11 that we give here is a consistency result improving a previous one of [3]:

Proposition 5.4 *If $X = \mathbf{\Omega}_3 X$, then $Con(X = \omega_3)$ holds.*

From [3] (Theorem 5.5. and Corollary 5.6) we have the weaker result that some particular fixed points of $\mathbf{\Omega}_3$ (e.g. $X \equiv \mathbf{Y}\mathbf{\Omega}_3$) can be consistently equated with ω_3, while here we deal with an arbitrary fixed point $X = \mathbf{\Omega}_3 X$ without knowing its exact shape. The idea is that even if the exact shape of X is unknown, the equation $X = \mathbf{\Omega}_3 X$ determines uniquely X^∞ as follows (where for typographical reasons we have omitted the "@" sign as a label of the binary nodes):

$X^\infty \equiv$

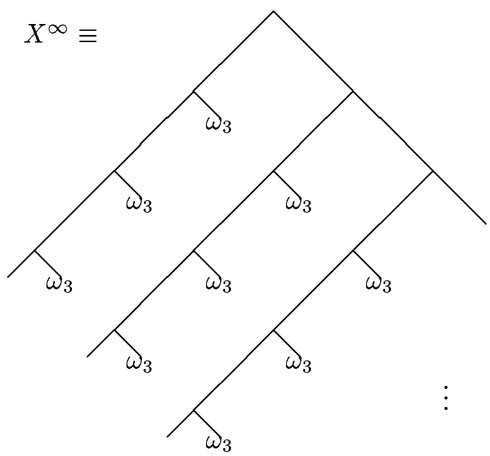

Figure 3

To prove the proposition we need a suitable confining class for ω_3 containing X^∞ and closed under the clauses of Definition 4.10. If we find such a class we will get the stronger result that all finite terms whose infinite normal form is in the class can be simultaneously equated to ω_3. Figure 3 suggests the following definition:

Definition 5.5 An infinite closed term U is an *ω_3-term* if every closed subterm T of U either is identical to ω_3 or it is an applicative term (as usual we regard finite terms as special cases of infinite terms).

So if we identify U with its parsing tree, then every node of U is either inside an occurrence of ω_3 or is labeled by the application node @. This includes the infinite normal form of $\mathbf{Y\Omega}_3$ and the infinite normal form of any X satisfying $X = \mathbf{\Omega}_3 X$. It also includes the infinite normal form of $\mathbf{YY}$, which contains only applicative subterms and no occurrences of ω_3. Notice that $\mathbf{YY}$ and $\mathbf{Y\Omega}_3$ are not themselves ω_3-terms, only their infinite normal forms are such.

Lemma 5.6 *For an infinite ω_3-term X, we have that X is a zero term iff X is an applicative term.*

Proof. Let X be an applicative ω_3-term. Then X is the application of two ω_3-terms A and B, say. To show that X is a zero term it suffices to show that every one-step β-reduct of AB is again an applicative ω_3-term. So let $AB \to C$. If the redex being contracted is AB itself, then A is an abstraction term and it follows that $A \equiv \omega_3$ and $C \equiv BBB$, which is obviously an applicative ω_3-term. If the redex being contracted is inside A or B then we apply the same argument inside A or B. QED

Lemma 5.7 *Let A be an infinite term arising from a finite one (so that A^∞ is well defined). If A is an ω_3-term, then A^∞ is also such.*

Proof. Assume that A is an ω_3 term. It is clear that any finite β-reduct of A is still an ω_3-term.
Claim: *for every k there exists a finite β-reduction $A \to^* A'$ such that every subterm of A' at depth $\leq k$ is a top normal form.*

To prove the claim, we argue by induction on k. Let $k = 0$. If A is an abstraction term, then $A \equiv \omega_3$ and there is nothing to prove. If A has the form $(BC)D$ with B, C ω_3-terms, then by Lemma 5.6 BC is a zero term, hence $A \equiv (BC)D$ is a top normal form. If A has the form $\omega_3 B$, then $A \to BBB$ and the previous case applies. If $k > 0$, observe first that every finite β-reduction starting from A does not create any abstraction term different from ω_3. Then use the argument given for the case $k = 0$, to every β-redex of level n. The claim follows.

As an immediate consequence any subterm of A^∞ is a top normal form, and therefore $\bot$ is not a subterm of A^∞. It remains to prove that A^∞ is an ω_3-term. Let D be a closed subterm of A^∞; we already know that D cannot be $\bot$ and that if the root of D is labeled by λx, then this λx arises from a finite β-reduction starting from A and therefore $D \equiv \omega_3$. So, if D is not ω_3, then it is an applicative term.QED

We are now ready to define our confining class:

Theorem 5.8 *Let $\mathcal{A}^\infty_{\omega_3}$ be the class of all applicative infinite terms which are ω_3-terms, are in normal form, and arise from a finite terms. Then $\mathcal{A}^\infty_{\omega_3}$ is a confining class of infinite terms for ω_3.*

Proof. By Lemma 5.6, we know that every element of $\mathcal{A}^\infty_{\omega_3}$ is a zero term. Now let $C^\infty[\,]$ be a non trivial context such that $C^\infty[B^\infty] \in \mathcal{A}^\infty_{\omega_3}$ and $B^\infty \in \mathcal{A}^\infty_{\omega_3}$; it is easy to see that $C^\infty[\omega_3]$ is an applicative ω_3-term. If it does not arise from a finite term there is nothing to prove since clause 2 of Definition 4.10 holds vacuously. If it does, then by Lemma 5.7 $(C^\infty[\omega_3])^\infty$ is an ω_3-term, and by Lemma 5.6 it is an applicative term. QED

We can now prove:

Corollary 5.9 *Let X be a finite closed term such that X^∞ is an ω_3-term. Then $Con(X = \omega_3)$ holds.*

Proof. If $X^\infty \equiv \omega_3$, then $X \to^* \omega_3$ and $Con(X = \omega_3)$ obviously holds. If $X^\infty \not\equiv \omega_3$, then X^∞ must be an applicative ω_3-term, hence it belongs to the confining class $\mathcal{A}^\infty_{\omega_3}$ therefore and $Con(X = \omega_3)$ holds. QED

Actually the method of the confining class proves more: all terms X such that X^∞ is an ω_3-term can be simultaneously equated to ω_3.

The line of reasoning we use to prove Theorem 5.9, can easily be extended to obtain a full generalization of the results of [3] concerning terms which can be written with only one variable.

6 Open Problems

6.1 Non-zero Terms

The method of confining class does not work for non-zero unsolvable terms. The main problem is that if we introduce a new reduction rule sending some non-zero terms to some fixed term, then an application of this rule may destroy β-redexes. It is however very likely that a similar technique, with suitable adaptations, could cope with non-zero terms. So we state the following problem.

Problem 6.1 Extend the confining class method to non-zero unsolvable terms.

Remark 6.2 The reader should observe that non-zero terms have an obvious way to give information about themselves, that is by acting on their arguments. As an exercise, the reader may check that $\lambda x.\mathbf{\Omega}(xx)$ is not easy (whilst $\mathbf{\Omega}N$ is easy for every N).

6.2 An Easiness Problem

Problem 6.3 Is $\mathbf{Y\Omega}_3$ easy ?

In our opinion, this is the most remarkable open problem on easy terms. This problem was posed ten years ago in [10] and, as far as we know, it is still open. In [3], we proved that $\mathbf{Y\Omega}_3$ is *normal form easy*, that is it can be consistently equated to every closed normal form. Moreover we proved $Con(\mathbf{Y\Omega}_3 = M)$ where M is *not* of the form: $\lambda x.xN_1N_2$, where at least one of N_1, N_2 is unsolvable and if N_i is solvable then $N_i \equiv x$. So a countexample, if any, has a very special form. In fact, this form is needed to break any confining class!

We illustrate this situation, by giving an example of of a term very similar to $\mathbf{Y\Omega}_3$ which is neverthless not easy.

We define the following combinators: $\mathbf{K}_1 \equiv \lambda xyz.xz$, $H \equiv \lambda x.xx\mathbf{K}_1$ and $\mathbf{\Omega}_{\mathbf{K}_1} \equiv HH$. Then $\mathbf{\Omega}_{\mathbf{K}_1} \to \mathbf{\Omega}_{\mathbf{K}_1}\mathbf{K}_1$, so $\mathbf{\Omega}_{\mathbf{K}_1}$ behaves like $\mathbf{\Omega}_3$ but with the normal form ω_3 replaced by the normal form $\mathbf{K}_1$. Both are self-similar zero terms of infinite order. We have:

Theorem 6.4 $\mathbf{Y\Omega}_{\mathbf{K}_1}$ *is not easy*

Proof. Let $N \equiv \lambda xyz.x(\mathbf{Y\Omega}_{\mathbf{K}_1}\mathbf{000}z)$ and consider the theory $\lambda\beta + \{\mathbf{Y\Omega}_{\mathbf{K}_1} = N\}$. This theory proves:

1. $N = \lambda xyz.x(N\mathbf{000}z) = \lambda xyz.xz = \mathbf{K}_1$;
2. $\mathbf{Y\Omega}_{\mathbf{K}_1} = N = \mathbf{K}_1$;
3. $\mathbf{\Omega}_{\mathbf{K}_1}N = \mathbf{\Omega}_{\mathbf{K}_1}(\mathbf{Y\Omega}_{\mathbf{K}_1}) = \mathbf{Y\Omega}_{\mathbf{K}_1} = N$;
4. $\mathbf{\Omega}_{\mathbf{K}_1}\mathbf{K}_1 = N$;
5. $\mathbf{\Omega}_{\mathbf{K}_1} = \mathbf{\Omega}_{\mathbf{K}_1}\mathbf{K}_1 = N$;
6. $NN = \mathbf{\Omega}_{\mathbf{K}_1}N = N$;
7. $\mathbf{K}_1\mathbf{K}_1 = \mathbf{K}_1$.

From the last equality we obtain $U = V$ for U and V arbitrary. QED

6.3 A consistency problem

Let X be a non-easy zero term, we would like to know what terms can be consistently equated to X. Infinite normal forms give a partial answer, since if Y behaves in a sufficiently similar way to X with respect to β-reduction, then $X^\infty = Y^\infty$ and $Con(X = Y)$ follows. It seems likely that the same holds for β-conversion. So, e.g., the following problem arises:

Problem 6.5 Let M be such that $M = M\omega_3$. Does $Con(M = \mathbf{\Omega}_3)$ hold?

Remark 6.6 M can very well be a normal form (see [5] for a general method for constructing normal forms with a prescribed "functional behaviour"). So, in the present case, infinite normal forms are of little use.

7 Appendix

Proof of Lemma 4.9. The proof is split into a series of sublemmas.

Lemma 7.1 $\to_{=c}$ *is Church-Rosser.*

Proof. Let $C \;{}_c\!\!\leftarrow A \to_c B$. We want to find a common c-reduct. Let $C_1[N_1]$ the c-redex reduced to N_1 in $A \to_c B$ and let $C_2[N_2]$ the c-redex reduced to N_2 in $A \to_c C$. So $C_1[\,]$ and $C_2[\,]$ are collapsing contexts. We can assume that the two c-redexes are nested (otherwise the result is trivial) and that the outer context coincides with A. So without loss of generality assume $C_1[N_1] \subseteq C_2[N_2] \equiv A$. We have to distinguish several cases. The main difficulty is that, in order to find a common c-reduct, we cannot simply collapse the two contexts $C_1[\,]$ and $C_2[\,]$ in a different order: in some cases we have to produce two new collapsing contexts different from those we started with. We treat the difficult cases first. In what follows let x and y be new variables not appearing in A, B or C.

Case 1. Suppose that $N_1 \subseteq N_2 \subseteq C_1[N_1] \subseteq C_2[N_2]$. Then we can write $C_2[N_2] \equiv U[V[Z[N_1]]]$ where $C_2[\,] \equiv U[V[\,]]$, $V[Z[\,]] \equiv C_1[\,]$ and $Z[N_1] \equiv N_2$. To find a common reduct it suffices to show that $U[\,]$ and $Z[\,]$ are collapsing contexts, for then we can write:

$$\begin{array}{ccc} U[V[Z[N_1]]] & \to_c & U[N_1] \\ \downarrow_c & & {}_c\!\!\downarrow^{*} \\ Z[N_1] & \to_c & N_1 \end{array}$$

Since $C_1[\,]$ and $C_2[\,]$ are collapsing, $V[Z[x]] \to^* x$ and $U[V[x]] \to^* x$. Thus $\lambda x.V[x]$ has both a right and a left inverse in the $\lambda\beta$-calculus and therefore it is β-convertible to the identity $\lambda x.x$ (see [1] Corollary 21.2.22). Since we already know that $U[V[x]] \to^* x$ and $V[Z[x]] \to^* x$, it follows that $U[x] \to^* x$ and $Z[x] \to^* x$ as desired.

Case 2. Suppose that N_1 and N_2 are disjoint and $N_2 \subseteq C_1[N_1] \subseteq C_2[N_2]$. Then we can write $C_2[N_2] \equiv U[V[N_1, N_2]]$ where:
(i) $C_2[y] \equiv U[V[N_1, y]] \to^* y$,
(ii) $C_1[x] \equiv V[x, N_2] \to^* x$.

We claim that this is impossible, so that case 2 actually never arises. We can assume that (i) and (ii) are head reductions (because head reductions are normalizing). The idea is to replace in (i) the term N_1 and all its residuals by a new variable x for as long as some residual of N_1 is applied to some other term.

If this never happens, then we can replace N_1 and all its residuals by a new variable x everywere in (i) and we get $U[V[x, y]] \to^* x$. It then follows that $N_2 = U[V[x, N_2]] = U[x]$ (by applying (ii) inside $U[\,]$). Thus $U[x]$ assumes the constant value N_2 for every x contradicting (i).

So some residual N_1' of N_1 is applied to some other terms in the reduction (i), namely we have a reduction of the form $U[V[N_1, y]] \to^* N_1' T_1 \dots T_n$ with

$n \geq 1$. If we take such a reduction of minimal length, then we can replace N_1' by x in this reduction and get a reduction of the form $U[V[x,y]] \rightarrow^* xQ_1 \ldots Q_n$ with $n \geq 1$. If we now substitute $y := N_2$, we reach a contradiction with (ii).

Case 3. Suppose $N_2 \subseteq N_1 \subseteq C_1[N_1] \subseteq C_2[N_2]$. Then we can write $C_2[N_2] \equiv U[C_1[Z[N_2]]]$ with $C_2[x] \equiv U[C_1[Z[x]]] \rightarrow^* x$, $C_1[x] \rightarrow^* x$ and $N_2 \equiv Z[N_1]$. It follows from the Church-Rosser property of β-reduction that $U[Z[x]] \rightarrow^* x$ and we can find a common c-reduct as follows:

$$\begin{array}{ccc} U[C_1[Z[N_2]]] & \rightarrow_c & U[Z[N_2]] \\ \downarrow_c & & \downarrow_c \\ N_2 & \equiv & N_2 \end{array}$$

Case 4. Suppose N_2 and $C_1[N_1]$ are disjoint (and contained in $C_2[N_2]$). Then we can write $C_2[N_2] \equiv U[N_2, C_1[N_1]]$ with $U[x, C_1[N_1]] \rightarrow^* x$ and $C_1[x] \rightarrow^* x$. It follows that $U[x, N_1] \rightarrow^* x$ and we use this collapsing context (plus an empty reduction) to find a common c-reduct.

Case 5. Suppose that $N_1 \subseteq C_1[N_1] \subseteq N_2 \subseteq C_2[N_2]$. This is easy: $C_1[\ ]$ and $C_2[\ ]$ can be collapsed in either order to yield the same c-reduct. QED

The next lemmas are essentially proved in [2] except that here we use an arbitrary weak confining class $\mathcal{C}$ (= the set of all μ-redexes) instead of the class of all mute terms.

Lemma 7.2 *Any diagram $C\ {}_\mu\!\leftarrow A \rightarrow_c B$ can be extended to a diagram of the form:*

$$\begin{array}{ccc} A & \rightarrow_c & B \\ \downarrow_\mu & & \downarrow_{=\mu} \\ C & \rightarrow_{=c} & D \end{array}$$

Proof. let U be the μ-redex contracted in $A \rightarrow_\mu C$ and let $H[N]$ be the c-redex of $A \rightarrow_c B$.

Case 1. Suppose $U \subseteq N$. This is easy: the two reductions can be performed in any order.

Case 2. Suppose that $N \subseteq U \subseteq H[N]$ with $N \not\equiv U$. We can then write $U \equiv U_1[N]$ and $H[N] \equiv H_1[U_1[N]]$ for some contexts $U_1[\ ]$ and $H_1[\ ]$ with one hole, such that $H_1[U_1[x]] \rightarrow^* x$, where x is a fresh variable. Since $U_1[N]$ is a μ-redex, it is in particular a zero term. Hence $U_1[x]$ is also a zero term. But then the only possibility to have $H_1[U_1[x]] \rightarrow^* x$ is that $U_1[x] \rightarrow^* x$ and $H_1[x] \rightarrow^* x$. It follows that N is a μ-redex because $U_1[N] \rightarrow^* N$ and μ-redexes are closed under β-reductions. Thus we have:

$$\begin{array}{ccc} H_1[U_1[N]] & \rightarrow_c & N \\ \downarrow_\mu & & \downarrow_\mu \\ H_1[M] & \rightarrow_c & M \end{array}$$

and the result follows.

Case 3. Suppose that $U \subseteq H[N]$ and U is disjoint from N. Then we can write $H[N] \equiv H_1[U, N]$ for some context $H_1[\ ,\ ]$ such that $H_1[U, x] \to^* x$. Since U is a μ-redex, U is a closed zero term, so U can be replaced by a variable in this reduction yielding $H_1[z, x] \to^* x$. Hence $H_1[M, x] \to^* x$ and $H_1[M, N] \to_c N$. Thus we have:

$$\begin{array}{ccc} H_1[U,N] & \to_c & N \\ \downarrow_\mu & & \equiv \\ H_1[M,N] & \to_c & N \end{array}$$

and the result follows.

Case 4. Suppose $H[N] \subseteq U$ and $H[N] \not\equiv U$. We can write $U \equiv U_1[H[N]]$. Since $U_1[H[N]]$ is a μ-redex and $U_1[H[N]] \to^* U_1[N]$, $U_1[N]$ is a μ-redex. Hence we have:

$$\begin{array}{ccc} U_1[H[N]] & \to_c & U_1[N] \\ \downarrow_\mu & & \downarrow_\mu \\ M & \equiv & M \end{array}$$

and we are done. QED

Lemma 7.3 $\to_{=\rho}$ *is Church-Rosser, and therefore* $\to_{=\rho}^*$ *is Church-Rosser.*

Proof. By the previous two lemmas and Lemma 4.7 QED

Lemma 7.4 *Any diagram* $C \ {}_c\!\leftarrow A \to_\beta B$*, can be extended to a diagram of the form:*

$$\begin{array}{ccc} A & \to_\beta & B \\ \downarrow_c & & {}_c\!\downarrow^* \\ C & \to_{=\beta} & D \end{array}$$

Proof. Let $\Delta \equiv (\lambda y.S)T$ be the β-redex contracted in $A \to_\beta B$, and let $H[N]$ be the c-redex of $A \to_c C$.

Case 1. Suppose that $\Delta \subseteq N$. By contracting Δ we obtain a reduction $N \to_\beta N'$. Thus we can write:

$$\begin{array}{ccc} H[N] & \to_\beta & H[N'] \\ \downarrow_c & & \downarrow_c \\ N & \to_\beta & N' \end{array}$$

which gives the desired result.

Case 2. Suppose that $N \subseteq \Delta \subseteq H[N]$ and $N \not\equiv \Delta$.

Notice that Δ cannot be of the form NT, because otherwise $H[N] \equiv H_1[NT]$ for some context $H_1[\ ]$ such that $H_1[xT] \to^* x$, which is impossible since T cannot be erased without erasing x as well. So N is contained either in S or in T. Suppose $N \subseteq T$. Then $\Delta \equiv (\lambda y.S)T_1[N]$ and $H[N] \equiv H_1[(\lambda y.S)T_1[N]]$ for some contexts $T_1[\ ]$ and $H_1[\ ]$ (with one hole) such that $H_1[(\lambda y.S)T_1[x]] \to^* x$. It then follows (form the unicity of normal forms for β-reduction) that $H_1[S[y := T_1[x]]] \to^* x$. In this reduction the left-hand-side may contain several occurrences of x, but only one of them has a residue (while the others are erased). This occurrence of x gives us a collapsing context and therefore we have:

$$\begin{array}{ccc} H_1[(\lambda y.S)T_1[N]] & \to_\beta & H_1[S[y := T_1[N]]] \\ \downarrow_c & & \downarrow_c \\ N & \equiv & N \end{array}$$

The case in which $N \subseteq S$ is similar.

Case 3. Suppose that $\Delta \subseteq H[N]$ and Δ is disjoint form N. Then $\Delta \subseteq H[\ ]$ and by contracting Δ we obtain a context $H'[\ ]$ with $H'[x] \to^* x$. The desired result follows.

Case 4. Suppose that $H[N] \subseteq \Delta$ and $\Delta \not\equiv H[N]$. Since $H[x] \to^* x$, if $H[N] \equiv \lambda y.S$, then the only possibility is that $H[N] \equiv N \equiv \lambda y.S$. In this case $A \to_c C$ is the empty reduction and there is nothing to prove.

So we can assume that $H[N]$ is contained either in S or in T. Suppose $H[N] \subseteq T$. Then $\Delta \equiv (\lambda y.S)T_1[H[N]]$ for some context $T_1[\ ]$ with one hole. So we can write:

$$\begin{array}{ccc} (\lambda y.S)T_1[H[N]] & \to_\beta & S[y := T_1[H[N]] \\ \downarrow_c & & {}_c\!\downarrow_* \\ (\lambda y.S)T_1[N] & \to_\beta & S[y := T_1[N]] \end{array}$$

The vertical reduction on the right is $\to_c^*$ rather than $\to_c$ because there are several occurrences of $H[\ ]$ being erased.

On the other hand if $H[N] \subseteq S$, then $H[N]$ being closed is not affected by the β-reduction and the diagram can be easily completed.

Case 5. If Δ and $H[N]$ are disjoint the result is trivial. QED

Lemma 7.5 *Any diagram $C\ {}_\mu\!\leftarrow A \to_\beta B$ can be extended to a diagram of the form $C \to_{=\beta} D\ {}^*_\mu\!\leftarrow B$.*

Proof. We can use the same argument of Theorem 4.3 (Claim 2), i.e. that a beta reduction can create many copies of, leave untouched or erase μ-redexes, but cannot make a μ-redex a non-redex . QED

Theorem 7.6 *$\to_\beta^*$ and $\to_\rho^*$ commute.*

Proof. We sketch the proof and leave details to the reader. Let r and s be, respectively, the multistep ρ- and β- reduction.To prove the theorem we argue by induction on the length of s (that we write $length(s)$).

Case 1. Let $length(s) = 1$. Then use Lemma 7.4, Lemma 7.5 and induction on the length of r.

Case 2. Now let $length(s) > 1$. Then use Case 1 to get a shorter β-reduction. QED

Corollary 7.7 $\rightarrow^*_{\beta\rho}$ *is Church-Rosser.*

Proof. As in Theorem 3.3, since $\rightarrow^*_\beta$ and $\rightarrow^*_\rho$ are a pair of commuting Church-Rosser notions of reduction we can conclude by the Hindley-Rosen Lemma . QED

The proof of Lemma 4.9 is now completed.

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References

[1] H. P. Barendregt, The Lambda Calculus, Studies in Logic 103, North-Holland, Amsterdam 1984.

[2] A. Berarducci, Infinite lambda-calculus and non-sensible models. Presented to the conference in honour of Roberto Magari, Siena 1994. Submitted for publication.

[3] A. Berarducci and B. Intrigila, Some new results on easy lambda-terms, Theoretical Computer Science 121 (1993) 71-88.

[4] C. Böhm, Alcune proprietà delle forme $\beta - \eta$-normali nel $\lambda - K$-calcolo, Pubblicazioni dell'Istituto per le Applicazioni del Calcolo, n. 696, Roma (19 pp.).

[5] C. Böhm and B. Intrigila, The Ant-Lion paradigm for strong normalization, Information and computation vol. 114, 1994, 30–49.

[6] N. Dershowitz, S. Kaplan, D. A. Plaisted, Rewrite, Rewrite, Rewrite, Rewrite, Rewrite, ..., Theoretical Computer Science 83, 1991.

[7] B. Intrigila, A problem on easy terms in λ-calculus, Fundamenta informaticae, vol. XV.1, (1991).

[8] G. Jacopini, A condition for identifying two elements of whatever model of combinatory logic, in C. Böhm (Editor) LNCS, vol. 37, Berlin-heidelberg-new York 1975.

[9] G. Jacopini and M. Venturini Zilli, Equating for Recurrent Terms of λ-calculus and Combinatory Logic, Quaderni dello IAC, serie III, N.85, Roma 1978.

[10] G. Jacopini and M. Venturini Zilli, Easy terms in the lambda-calculus, Fundamenta informaticae, vol. VIII.2, (1985)

[11] R. Kennaway, J. W. Klop, R. Sleep, F-J de Vries, Transfinite reductions in orthogonal term rewriting systems, preprint 1993, to appear in: Information and computation.

[12] R. Kennaway, J. W. Klop, R. Sleep, F-J de Vries, The infinite lambda calculus, manuscript 1994.

[13] G. Mitschke, λ-Kalkül, δ-conversion und axiomatische Rekursionstheorie, Preprint Nr. 274, Technische Hochschule, Darmstadt. Fachbereit Mathematik, 77 pp.