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SPIN AND PIN$^-$ STRUCTURES, IMMERSED AND EMBEDDED SURFACES AND A RESULT OF SEGRE ON REAL CUBIC SURFACES

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0. INTRODUCTION

THE initial motivation of this paper comes from a result of Segre [12] about the real lines on a real cubic surface. As it is well known a smooth complex cubic surface has exactly 27 (complex) lines. In the real case this is not always true anymore. A smooth real cubic surface can have 27, 15, 7 or 3 real lines. This has been well known since the 19th century. The result of Segre we are alluding to is far less known and introduces a more subtle difference between the real and complex cases. Segre distinguishes two types of real straight lines (see below Section 6 for precise definitions) and shows that on a real cubic surface with 27 real lines 15 are of one type and 12 of the other (in fact the result is more complete and gives the classification in all cases—see Theorem 6.2 below). Segre proved this result by studying the degeneration of non-singular cubic surfaces to the union of 3 planes and a special "graphical" way of representing the occurring situations.

Noting that the basic difference between the two types of lines is that their respective tubular neighbourhoods in the surface differ by a *full twist* in $\mathbb{P}^3$, our initial aim was to give a new interpretation and a new proof of this result in terms of the Pin$^-$ structure induced by the embedding of the surface in $\mathbb{P}^3$ ($\mathbb{P}^3$ taken with a fixed Spin structure). More precisely, we will show that the two type of lines distinguished by Segre are also differentiated at the homology level by the mod 4 quadratic form canonically associated with the above Pin$^-$ structure.

A further point of interest is that, assuming that the complexification $X(\mathbb{C}) \subset \mathbb{P}^3(\mathbb{C})$ of the surface X is also non-singular and that the surface is an M-surface, there is another Pin$^-$ structure, induced by the embedding of $X(\mathbb{R})$ in $X(\mathbb{C})$ (see [6]). This second form differs from the first by a "*privileged*" class in $\mathrm{H}^1(X, \mathbb{Z}/2)$, a class which seems to deserve further investigations. We will explicitly compute this class for quadric and cubic surfaces.

The work done for surfaces in $\mathbb{P}^3$ led us to study more generally immersions of surfaces in arbitrary orientable 3-manifolds. Using the theory of Spin and Pin$^-$ structures (see, for example, [10] and the book [6]), we consider the problem of associating, as above, quadratic forms with immersions of surfaces in 3-manifolds. We have done this by reformulating results of Pinkall [11], where only the case of $\mathbb{R}^3$ is considered and results of Hass and Hughes [8] where the immersions of surfaces into arbitrary 3-manifolds is studied, but not in terms of quadratic forms. Following Pinkall we will also introduce the notion of immersed surfaces (an equivalence class of immersions—see Sections 4 and 10) and study the relationships between different equivalence relations on immersed surfaces (regular homotoppy, cobordism, equivalence of the Pin$^-$ structures) extending the results Pinkall has obtained for $\mathbb{R}^3$. In Section 11 we apply the results of this analysis to compute the semi-group $N_2(M)$ of cobordism classes of immersed surfaces in an arbitrary orientable 3-manifold M and prove that in fact it is a group. This is probably the main result of this paper.

1. QUADRATIC FORMS MOD 4

For a reference on this part see, for example, [6, pp. 99–101] or the Appendix of [1]. We will only recall the basic facts we are going to use and, where needed, sketch some proofs.

Let V be $\mathbb{Z}/2$-vector space with a non-degenerate symmetric bilinear form $(.)$. We will say that $q: V \to \mathbb{Z}/4$ is a *quadratic form* mod 4 *associated with* $(.)$ if for all x and y in V we have

$$q(x + y) = q(x) + q(y) + 2(x.y) \tag{1.1}$$

where 2 is the only non-trivial morphism from $\mathbb{Z}/2$ to $\mathbb{Z}/4$.

Such a triple $(V, (.), q)$ will be called a *quadratic space*. We will say that two quadratic spaces $(V, (.), q)$ and $(V', (.)', q')$ are *isometric* or more briefly that two quadratic forms q and q' are *isometric* if there exists an isomorphism $l: V \to V'$ such that $q' = q \circ l$ (note that by the definition this will ensure that the two bilinear forms are also isometric).

We recall the following facts on such spaces.

First note that $q(0) = 0$ and that $q(x) \equiv (x.x) \bmod 2$.

If $\dim V = 1$ and $(.)$ is the only non-trivial bilinear form, then there are 2 associated quadratic forms, one q_+ for which $q_+(e) = 1$ (where e is the generator of V) and the other q_- for which $q_-(e) = -1$.

If $\dim V = 2$ and the bilinear form is defined, in terms of a basis $\{e_1, e_2\}$, by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

then there are 4 associated quadratic forms: Q_1 is defined by $Q_1(e_1) = Q_1(e_2) = 0$ and $Q_1(e_1 + e_2) = 2$, Q_2 is defined by $Q_2(e_1) = Q_2(e_2) = Q_2(e_1 + e_2) = 2$ and the other two are isometric to Q_1 and correspond to permutations of e_1, e_2 and $e_1 + e_2$.

A quadratic space is said *isotropic* (or *even*) if for all $x \in V$, $(x, x) = 0$, or equivalently $q(x) \equiv 0 \bmod 2$. In such a case there exists an ordinary quadratic form $\bar{q}: V \to \mathbb{Z}/2$ such that $q = 2\bar{q}$ (2 being as before the morphism from $\mathbb{Z}/2$ to $\mathbb{Z}/4$).

A quadratic space $(V, (.), q)$ is called *neutral* if V contains a subspace H such that $2 \dim H = \dim V$ and $q = 0$ on H.

Taking orthogonal sums induces a semi-group structure on the space of quadratic spaces and on the space of neutral quadratic spaces, since, obviously, the orthogonal sum of neutral spaces is again neutral. The quotient of these two semi-groups is actually a group and is called the *Witt group of quadratic forms* mod 4 and denoted $\mathrm{WQ}(\mathbb{Z}/2, \mathbb{Z}/4)$. We will say that two quadratic spaces, or two quadratic forms, are *Witt equivalent* if they represent the same class in the Witt group.

To Witt equivalence is associated an important invariant that we define as follows. Let, for x in V,

$$\Phi(x) = \exp\left(\frac{i\pi q(x)}{2}\right) = i^{q(x)}$$

and let

$$A(q) = 2^{-\dim V/2} \sum_{x \in V} \Phi(x). \tag{1.2}$$

The complex number $A(q)$ is called the *Arf–Brown* invariant of q or of the quadratic space $(V, (.), q)$.

Remark 1.3. (i) If the quadratic space is isotropic, then, as noted above, $q = 2\bar{q}$. In this case $A(q)$ coincides with the usual Arf invariant of $\bar{q}$.

(ii) Elementary computations show that

$$A(q_+) = \frac{1+i}{\sqrt{2}}, \qquad A(q_-) = \overline{A(q_+)}, \qquad A(Q_1) = 1, \qquad A(Q_2) = -1.$$

More generally one can show that $A(q) = 1$ if the form q is neutral.

THEOREM 1.4. *The map, $A:(V,(.),q) \mapsto A(q)$, induces an isomorphism between the Witt group $WQ(\mathbb{Z}/2, \mathbb{Z}/4)$ and μ_8, the group of 8th roots of unity.*

Proof (Outline). Elementary computations show that

(i) $A(q_1 + q_2) = A(q_1) \cdot A(q_2)$.

(ii) $4q$ (where by $4q$ we mean $q \perp q \perp q \perp q$) is isometric to $4(-q)$. This implies in particular that $8q$ is always neutral.

(iii) $q_+ \perp q_-$ is neutral and non-isotropic. As a consequence every quadratic form q is Witt equivalent to a non-isotropic one, namely $q \perp q_+ \perp q_-$.

We also note the well-known fact that if $(V,(.),q)$ is not isotropic then, there exists an orthogonal sum decomposition of V into spaces of dimension 1. From this, and (ii) and (iii) above, it is easy to prove that WQ($\mathbb{Z}/2, \mathbb{Z}/4$) is cyclic of order a divisor of 8 and generated by q_+.

To end the proof we only need to note, since, by (i), the map is a morphism, that $A(q_+) = (1 + i)/\sqrt{2}$ is a primitive 8th root of unity. □

Theorem 1.4 and its proof have some important consequences that we will use in the sequel.

1.5. (i) Two spaces $(V_1, (.)_1, q_1)$ and $(V_2, (.), q_2)$ are isometric if and only if the following conditions are satisfied:

— $\dim V_1 = \dim V_2$;

— $(.)_1$ and $(.)_2$ are either both isotropic or both non-isotropic;

— $A(q_1) = A(q_2)$.

Note in particular that if $(V_1, (.)_1)$ and $(V_2, (.)_2)$ are equal, then q_1 and q_2 are isometric if and only if the Arf–Brown invariants are the same. Note also, that one can replace the second condition above by,

— The mod 2 reductions of q_1 and q_2 are either both zero or both non-zero.

(ii) $A(q)^4 = (-1)^{\dim V}$ and hence $A(q)$ determines $\dim V \bmod 2$.

To end on quadratic forms mod 4, note that the fact that there are 2 if the dimension is 1 and 4 if the dimension is 2 (and the bilinear form fixed of course) generalizes. We have the following lemma.

LEMMA 1.6. *If V is a $\mathbb{Z}/2$-vector space with a non-degenerate symmetric bilinear form* (.) *then there are* $2^{\dim V}$ mod 4 *quadratic forms associated with* (.). *If q is one, then the others are of the form*

$$q'(x) = q(x) + 2(u.x) = q(x) + 2\ell(x)$$

for some u in V (respectively a linear form ℓ on V).

The proof is easy.

2. SPIN STRUCTURES ON 3-MANIFOLDS AND PIN⁻ STRUCTURES ON SURFACES

The content of this section is again classical (we will use the paper of Kirby and Taylor [10] and Kirby [9, Ch. IV] as basic references).

To discuss the results we have to recall that the circle S^1 has 2 Spin_1 structures. In particular, a SO_1-bundle (which is necessarily trivial) on S^1 has two Spin_1-structures. One corresponds to the trivial double covering of S^1 by two copies of S^1 and the other to the double covering of S^1 by S^1. Following [9] we will call the first canonical (it corresponds to the unique trivialization of the SO_1-bundle).

An SO_2-bundle on S^1 also has 2 Spin_2-structures. These are best interpreted in terms of framings. Consider a framing of the normal bundle to a circle in $\mathbb{R}^3$. Adding a tangent vector to the circle yields a framing of the restriction of the tangent bundle of $\mathbb{R}^3$. Then one can describe the two Spin_2 structures as follows. One corresponds to framings that do not extend to framings of the restriction of the tangent bundle of $\mathbb{R}^3$ to the disk bounded by S^1 (see Fig. 1) and the other to ones that do extend (see Fig. 2).

Two framings correspond to the same Spin structure if they differ by an even number of turns along S^1 (see, for example, [10] or [9]).

LEMMA 2.1. *If $\xi = \xi_1 \oplus \xi_2$ is a direct sum decomposition of bundles, then a* Spin *structure on two of the bundles determines a* Spin *structure on the third.*

See [9, Proposition 3, p. 37].

For Pin^- structures on surfaces the basic result we will need is the following proposition.

PROPOSITION 2.2. *There is a canonical one to one correspondence between* Pin^- *structures on a surface F and* mod 4 *quadratic forms on* $\mathrm{H}_1(F, \mathbb{Z}/2)$ *associated with the intersection form.*

See [10, Theorem 3.2].

To define a mod 4 quadratic form associated with the intersection form (and hence a Pin^- structure) it is in fact enough to define a function $\hat{q}$ that assigns an element in $\mathbb{Z}/4$ to

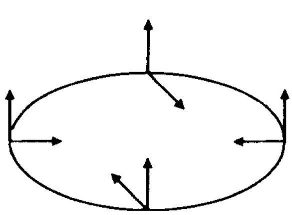

Fig. 1.

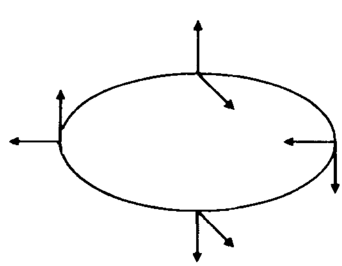

Fig. 2.

each embedded disjoint union of circles in F subject to the following conditions (see [10, Lemma 3.4].):

2.3. (a) $\hat{q}$ is additive on disjoint unions; that is, if L_1 and L_2 are two embedded disjoint unions and $L_1 \amalg L_2$ is again embedded, then $\hat{q}(L_1 \amalg L_2) = \hat{q}(L_1) + \hat{q}(L_2)$.

(b) If K_1 and K_2 are two circles that cross transversely at r points then replacing each crossing we get an embedded disjoint union L. We must have $\hat{q}(L) = \hat{q}(K_1) + \hat{q}(K_2) + 2r$.

(c) If K is an embedded circle that bounds a disk in F, then $\hat{q}(K) = 0$.

Note that to define $\hat{q}$ we only need to define it on embedded circles and extend it to disjoint unions by (a).

3. THE PIN$^-$ STRUCTURE OF AN IMMERSED SURFACE IN A SPUN 3-MANIFOLD

In this section we review results of Pinkall [11] and Hass and Hughes [8] and adapt them for further use.

Let M be an orientable 3-manifold (smooth and without boundary) and let F be a surface (smooth, compact and without boundary but not necessarily orientable: for simplicity, we also assume that the surface is connected). We fix once and for all an orientation on M and when we speak of M we will mean M with this fixed orientation. As is well known M always admits a Spin structure. In general, there are more than one since there is a simply transitive action of $\mathrm{H}^1(M, \mathbb{Z}/2)$ on the space of Spin structures of M (cf. Remark 3.5). Let Θ be one. We will say that (M, Θ) is a spun manifold.

Let $f: F \to M$ be an immersion. We are going to associate with f and Θ, a Pin$^-$ structure, $\Pi_{f,\Theta}$, on F.

To do this let f^*T_M be the pull-back of the tangent space T_M to M. Since M is oriented, the normal bundle N_f to F in f^*T_M is isomorphic to the determinant bundle Δ_F of T_F, the tangent bundle to F. We then have an identification between f^*T_M and $T_F \oplus \Delta_F$.

The action Θ induces a Spin structure on f^*T_M. But by [10, Lemma 1.7, p. 187], there is a one to one correspondence between Spin structures on $T_F \oplus \Delta_F$ and Pin$^-$ structures on T_F and hence on F. This defines $\Pi_{f,\Theta}$.

In order to get a better understanding of $\Pi_{f,\Theta}$ we are going to describe it in terms of a mod 4 quadratic form $q_{f,\Theta}$.

Let K be an embedded circle in M. The action Θ allows us to select a class of even framings on K as follows: Θ induces a Spin structure on $T_{M|K}$. We have a direct sum decomposition $T_{M|K} = N_{K|M} \oplus T_K$. Take on T_K the canonical Spin structure. Then by Lemma 2.1 we get a well-defined Spin structure on the normal bundle $N_{K|M}$. We will call *odd* the class of framings associated with this structure. We will call *even* the other class (so that, for example, the framing shown in Fig. 1 is even).

Now let C be an embedded circle in F such that $f_{|C}$ is an embedding. Define $\hat{q}_{f,\Theta}(C)$ to be the number mod 4 of left half turns (positive half turns) which the normal bundle to F restricted to C, $N_{F|M|C}$, does when moving along C with respect to any even framing (note that this does not depend on which way we move along C).

LEMMA 3.1. *The number $\hat{q}_{f,\Theta}(C)$ only depends on the class of C in $\mathrm{H}_1(F, \mathbb{Z}/2)$. Moreover, the induced map $q_{f,\Theta}: \mathrm{H}_1(F, \mathbb{Z}/2) \to \mathbb{Z}/4$ is a quadratic form* mod 4 *associated with the intersection form.*

Since our construction is exactly the one which associates with a Pin$^-$ a quadratic form mod 4 (see Proposition 2.2), the proof of Lemma 3.1 is the same as the proof of Theorem 3.2 of [10] (see in particular the proof of Lemma 3.4).

LEMMA 3.2. *For a fixed choice of Θ, $q_{f,\Theta}$ only depends on the regular homotopy class of f.*

Pinkall [11] proved Lemma 3.2 in the special case $M = \mathbb{R}^3$. Actually, it is quite easy to see that his proof extends to our general situation. The essential reason is that if $Tub_{K|F}$ is a tubular neighbourhood of K in F, then $\hat{q}_{f,\Theta}(K)$ only depends on the regular homotopy class of f restricted to $Tub_{K|F}$ and this is easy to see.

Let ξ be a homotopy class of maps from F into M and let $Imm_\xi(F,M)$ be the set of *regular homotopy classes of immersions of F into M belonging to* ξ (note that $Imm_\xi(F,M)$ is non-empty—see, for example, [8, Lemma 1.2]). Also let $\mathcal{Q}_F$ be the set of quadratic forms mod 4 on $\mathrm{H}_1(F, \mathbb{Z}/2)$ associated with the intersection form.

By Lemmas 3.1 and 3.2 (and also Proposition 2.2) we have defined a map

$$q_\Theta : Imm_\xi(F,M) \to \mathcal{Q}_F \cong \{\mathrm{Pin}^- \text{ structures on } F\}. \tag{3.3}$$

THEOREM 3.4. *The map q_Θ defined in* (3.3) *is a bijection.*

This theorem is a unified formulation of the main results of Pinkall [11] and of Hass and Hughes [8]. Pinkall only considers the situation when $M = \mathbb{R}^3$ and proves that in this case the quadratic form only depends on the regular homotopy class of the immersion f and conversely that if $q_{f,\Theta} = q_{f',\Theta}$, then f and f' are regularly homotopic. But for $M = \mathbb{R}^3$ there is only one Spin structure (for a fixed orientation) and all immersions are homotopic to each other. Hence, for $M = \mathbb{R}^3$, Theorem 3.4 reduces to the result of Pinkall.

The result of Hass and Hughes is reformulated in the following.

Proof of Theorem 3.4 (Outline). For any simple closed curve C embedded in the surface F, Hass and Hughes introduce the notion of *adding a kink along C* (see [8, pp. 104–105] for an explicit description). This changes the immersion f in a tubular neighbourhood of C but leaves it unchanged elsewhere; moreover, the new immersion g obtained in this fashion is in the same homotopy class as f. It is quite easy, following the explicit description given in [8], to see that this operation corresponds to adding locally a *full twist* along all curves that intersect C transversely.

Now the action of $\mathrm{H}^1(F,\mathbb{Z}/2)$ on $Imm_\xi(F,M)$ can be described as follows. Let $c \in \mathrm{H}^1(F,\mathbb{Z}/2)$, then f_c is obtained from $f_0 = f$ by adding a kink along C, where C is an embedded circle in F with dual class c in $\mathrm{H}^1(F, \mathbb{Z}/2)$ such that $f_{|C}$ is an embedding. The main result of [8] is that this action is simply transitive or in other words the map $c \mapsto f_c$ is bijective. This is not an elementary result as the proof of the surjectivity uses in an essential way the Hirsh–Smale theorem (but one should note that in the special case of a connected surface punctured in a point there is an elementary and self-contained proof—see the appendix of [6, p. 114]).

On the other hand, we have an action of $\mathrm{H}^1(F,\mathbb{Z}/2)$ on $\mathcal{Q}_F$ defined by

$$q_c(\alpha) = q(\alpha) + 2c(\alpha) = q(\alpha) + 2(c^*.\alpha) \qquad (\mathrm{mod}\, 4).$$

By Lemma 1.6 this action is again simply transitive.

Recalling the description of the quadratic forms in terms of *half twists* and the above discussion, it is easy to check that the two actions correspond and that the map q_Θ is a bijection. □

Remark 3.5. The map $q_{f,\Theta}$ depends on Θ as follows: $\mathrm{H}^1(M, \mathbb{Z}/2)$ acts transitively on $Spin(M)$ (the space of Spin structures). If Θ' differs from Θ by $c \in \mathrm{H}^1(M, \mathbb{Z}/2)$, then a framing of an embedded cricle K is Θ'-even if and only if either K is Θ-even and $c(K) = 0$ or K is Θ-odd and $c(K)$ is non-zero. Hence, for every $\alpha \in \mathrm{H}_1(F, \mathbb{Z}/2)$,

$$q_{f,\Theta'}(\alpha) = q_{f,\Theta}(\alpha) + 2f^*(c)(\alpha) \qquad (\mathrm{mod}\, 4).$$

We end this section with a few words on how to compute $\hat{q}_{f,\Theta}$ (but see Section 7 for a further discussion).

First we note that, for a general M, if K is an embedded circle and defines a trivial class in $\mathrm{H}_1(M, \mathbb{Z}/2)$, then $\hat{q}_{f,\Theta}(K)$, does not depend on the choice of the Spin structure Θ. In this case $\hat{q}_{f,\Theta}(K)$ is just the linking number mod 4 of K with the boundary $\partial Tub_{K|F}$ of an embedded tubular neighbourhood of K in F (this is well defined since the class of K is trivial); see also [10, p. 209].

If the class of K is not trivial the computation of $\hat{q}_{f,\Theta}(K)$ is more complicated but in the case $M = \mathbb{P}^3(\mathbb{R})$ we can give an intuitive way of computing this number.

For this we will need to introduce some notations. Consider $\mathbb{P}^3(\mathbb{R})$ as a compactification of $\mathbb{R}^3$. As before, we assume we have fixed an orientation on $\mathbb{P}^3(\mathbb{R})$, and hence one on $\mathbb{R}^3$. Fix the Spin structure Θ on $\mathbb{P}^3(\mathbb{R})$ such that $q_{j,\Theta} = q_+$ (see Section 1), where j is the canonical embedding of $\mathbb{P}^2(\mathbb{R})$ in $\mathbb{P}^3(\mathbb{R})$. In the sequel we will assume, if not specified explicitly otherwise, that this is the Spin structure we consider on $\mathbb{P}^3(\mathbb{R})$.

Now let $Tub_{K|F}$ be a tubular neighbourhood of K in F embedded by f. Let P_0 be the plane at infinity in $\mathbb{P}^3$, we may always assume, deforming f slightly if necessary, and that in some neighbourhood U of P_0, $Tub_{K|F} \cap U$ is contained in a plane P transverse to P_0. Now we can count the number of left half turns which the normal bundle to F, restricted to K, does when moving along K outside of U. We define $\hat{q}_{f,\Theta}$ to be, this number $+1$, taken as mod 4.

Remark 3.6. Since $\mathrm{H}^1(\mathbb{P}^3(\mathbb{R}), \mathbb{Z}/2) = \mathbb{Z}/2$, $\mathbb{P}^3$ has exactly, for a fixed orientation, two Spin structures, the second one corresponds to replacing q_+ by q_- in the above construction. But if h is the generator of $\mathrm{H}^1(\mathbb{P}^3(\mathbb{R}), \mathbb{Z}/2)$ then $j^*(h)$ is the class of a line in $\mathbb{P}^2$. Hence, if we had taken this other Spin structure on $\mathbb{P}^3(\mathbb{R})$ we would have added -1 in place of $+1$ (see Remark 3.5).

4. IMMERSED AND EMBEDDED SURFACES IN A 3-MANIFOLD

We will need to introduce quite a few notations and definitions.

Following Pinkall [11] we will say that two immersions of a surface, f and g, are equivalent if there exists a diffeomorphism φ of F such that $g = f \circ \varphi$. We will denote by $[f]$ the class of f under this equivalence relation and call $[f]$ an *immersed surface of type F* in M (or simply an immersed surface if there is no ambiguity). An *embedded surface* is an immersed surface $[f]$, with f an embedding. We will say that $[f]$ and $[g]$ are homotopic (resp. regularly homotopic) if f is homotopic (resp. regularly homotopic) to $g \circ \varphi$ for some diffeomorphism φ.

Again following Pinkall [11, Section 6], we introduce the following definition.

Definition 4.1. Let $[f]$ and $[g]$ be two immersed surfaces in M of types F_1 and F_2, respectively (F_1 and F_2 not necessarily of the same topological type or necessarily connected). We will say that $[f]$ and $[g]$ are *cobordant* immersed surfaces if there exist a 3-manifold X, having as boundary the disjoint union of F_1 and F_2, and an immersion $h: X \to M \times [0, 1]$ such that h is transverse to $M \times \{0, 1\}$ and $f \times \{0\} = h|F_1$ and $g \times \{1\} = h|F_2$.

Note that the above definition makes sense, i.e. only depends on the classes and not on the representatives f and g. This follows from the fact that if φ is a diffeomorphism of F and f an immersion of F, then f is cobordant to $f \circ \varphi$ by means of the 3-manifold X obtained from $F \times [0, 1]$ and $F \times [1, 2]$ by gluing, via φ^{-1}, two copies of $F \times \{1\}$.

PROPOSITION 4.2. *Two embedded surfaces* $[f]$ *and* $[g]$ *are cobordant if and only if they have same class in* $\mathrm{H}_2(M, \mathbb{Z}/2)$. *In particular, two homotopic embedded surfaces of the same type are cobordant.*

Proof. If $[f]$ and $[g]$ are cobordant then they clearly represent the same class.

Let L be a line bundle on M associated with the class in $\mathrm{H}^1(M, \mathbb{Z}/2)$ Poincaré dual to the fundamental class of $[f]$. In fact we can construct L with a section S such that S is transverse to the zero section Z of L and $[f] = S \pitchfork Z$. For $[g]$ we can also find L' and S' with the same properties. If $[g]$ is in the same class as $[f]$ then L and L' are isomorphic. Using this isomorphism we can in fact assume that $L = L'$. Consider the pullback L^* of such a line bundle to $M \times [0, 1]$, via the natural projection onto M. For $t \in [0, 1]$ small enough we can build a section R such that it coincides with $S \times \{x\}$ on $M \times [0, t]$, with $S' \times \{x\}$ on $M \times [1 - t, 1]$. The transverse intersection of R and Z^* realizes the required cobordism. □

Let $[f]$ be an immersed surface in a spun manifold (M, Θ). Clearly, the isometry class of the quadratic form $q_{f,\Theta}$ does not depend on the choice of the representative f; hence, with $[f]$ we can associate a well-defined eighth root of unity, the Arf–Brown invariant of $q_{f,\Theta}$. We will call this *the Arf–Brown invariant of* $[f]$.

We will need to extend this notion to immersions of disjoint union of surfaces. This we do by defining, in this case, the invariant as the product of the invariants of the different components.

PROPOSITION 4.3. *If* $[f]$ *and* $[g]$ *are cobordant immersed surfaces, in a spun 3-manifold* M, *then they have the same Arf–Brown invariant. In particular if they are of the same topological type, the quadratic forms* $q_{f,\Theta}$ *and* $q_{g,\Theta}$ *are isometric.*

Proof. Let X and $h: X \to M \times [0, 1]$ be the 3-manifold and map realizing the cobordism and let $t: M \times [0, 1] \to [0, 1]$ be the projection. Without loss of generality, we may assume that $t \circ h$ is a Morse function. Then considering all the possible accidents when passing through a critical point of $t \circ h$, one checks that the Arf–Brown invariant does not change (see [11, pp. 432–433]). The last statement follows immediately from 1.5.

A more conceptual argument using more deeply the fact that Witt-equivalence is the algebraic counterpart to cobordism would run as follows.

If an immersed surface F is cobordant to zero, the quadratic form vanishes on the kernel of the morphism $j_*: \mathrm{H}_1(F, \mathbb{Z}/2) \to \mathrm{H}_1(X, \mathbb{Z}/2)$ $(j: F \subsetneq X)$ and the quadratic form is in fact neutral (cf. [6, pp. 109, 111] and following). Hence, the Arf–Brown invariant is equal to 1.

5. REVIEW OF SOME FACTS ON COMPLEX CUBIC SURFACES

We recall here the classical facts about cubic surfaces that we will need in the sequel. There are many references for these and from the point of view we are going to use one can look, for example, at [7] or [5].

A smooth, complex, cubic surface in $\mathbb{P}^3(\mathbb{C})$ is isomorphc to $\mathbb{P}^2(\mathbb{C})$ blown up in six points $p_1, \dots, p_6$ in general position, where by general position we mean that no three points are on a line and the six points are not on a conic. Conversely, any surface obtained in this way by blowing up $\mathbb{P}^2$ in six points in general position can be embedded in $\mathbb{P}^3$ as a smooth cubic surface (such an embedding is given by the anticanonical line bundle which here corresponds to the linear series of cubic curves passing through the six points).

A cubic surface has 27 (complex) lines. These are

5.1

— the six exceptional lines $E_1, \dots, E_6$ obtained by blowing up $p_1, \dots, p_6$;
— the 15 D_{ij}'s pullback of the 15 lines passing through two of the p_i's;
— the six C_j's pullback of the six conics passing through five of the p_i's ($i \neq j$).

If X is the cubic surface then, $\mathrm{H}_2(X, \mathbb{Z}) \cong \mathbb{Z}^7$ is generated by $e_1, \dots, e_6$ and l where the e_i's are the fundamental classes of the E_i's and l is the class of the pullback of a general line in $\mathbb{P}^2$. In terms of these generators the classes of the 27 lines are the following:

5.2

— the e_i's;
— the classes $l - e_i - e_j$, $i \neq j$;
— the classes $2l - \sum_{k=1}^{5} e_{i_k}$, $i_k \neq i_{k'}$ for $k \neq k'$.

Note also that the class of a hyperplane section, or if one prefers the class of the anticanonical divisor, is $-K = 3l - \sum e_i$.

In fact, the construction provides even more information since we also obtain the intersection form on $\mathrm{H}_2(X, \mathbb{Z})$ by noting that

— $(e_i.e_j) = 0$ if $i \neq j$ and -1 if $i = j$;
— $(e_i.l) = 0$;
— $(l.l) = 1$.

Recalling that the degree of a curve on X is just the intersection number of this curve with the hyperplane section, it is easy to prove that the 27 classes described in 5.2 are exactly those with self-intersection -1 and degree 1 (hence lines).

6. REAL CUBIC SURFACES AND THE RESULT OF SEGRE

From the construction given in Section 5 it is easy to see that if we blow up six real points of $\mathbb{P}^2$ we obtain a real cubic surface. Moreover, such a surface will have 27 real lines.

Of course, there are other ways to obtain real cubic surfaces. One can blow up four real points and a pair of complex conjugate points, or two real points and two pairs of complex conjugate points, etc. There is also an other way which yields a surface with two connected components (for a complete list of real cubic surfaces and a discussion of the different cases, see, for example, [13] or [3] or, of course, Segre [12]). In the following, for clarity, we will essentially concentrate on the real cubic surface with 27 real lines but the other cases can be described as follows. Let U_k be the non-orientable topological surface with Euler characteristic $1 - k$. Then for the real part $X(\mathbb{R})$ of a smooth real cubic surface X, we have the following possibilities.

6.1. $X(\mathbb{R})$ is homeomorphic to

(i) U_6 and the surface has 27 real lines;
(ii) U_4 and the surface has 15 real lines;
(iii) U_2 and the surface has seven real lines;
(iv) $U_0 \cong \mathbb{P}^2$ and the surface has three real lines;
(v) $\mathbb{P}^2 \amalg S^2$ and the surface has again three real lines all contained in the component homeomorphic to $\mathbb{P}^2$.

Now let X be a smooth real cubic surface and let D be one of its real lines. Let P be a plane in $\mathbb{P}^3$ containing D. Since the intersection of P with X must be of degree 3 we must have $P \cap X = D \cup C$ where C is a conic (eventually degenerated). It is easy-to see that in fact P is tangent to X at the points of intersection of D and C and that conversely the tangent plane to X at a point $x \in D$ must contain D. This means that we have a pencil of conics associated with D and that this pencil defines an involution on D (sending one point of intersection of C and D to the other). Obviously, such an involution is real (sends the real part to the real part). In particular, it restricts to an involution of $D(\mathbb{R})$. But $D(\mathbb{R}) \cong \mathbb{P}^1(\mathbb{R}) \cong S^1$ and there are two types of involutions on S^1. One type has no fixed points and the other has two fixed points. Segre calls the line D *elliptic* in the first case and *hyperbolic* in the second. His result is the following theorem.

THEOREM 6.2. (Segre [12]). *If X is a smooth real cubic surface, then*

— 12 *lines are elliptic and* 15 *hyperbolic if X is of type* 6.1 (i);
— *six are elliptic and nine hyperbolic if X is of type* 6.1 (ii);
— *two are elliptic and five hyperbolic if X is of type* 6.1 (iii);
— *all three are hyperbolic in cases* 6.1 (iv) *and* (v).

One should note that the first inerpretation of Theorem 6.2 that comes into mind, i.e. that in the first case the 12 elliptic lines correspond to $E_1, \ldots, E_6$ and the six C_j's and the 15 hyperbolic lines to the D_{ij}'s (see 5.1), is false (that is, this can be the case—see 8.1(i) and the construction in Section 8—but in general it is not so). To see why this is, assume that the above interpretation is correct. The lines $E_1, E_2, E_3, D_{45}, D_{46}$ and D_{56} do not intersect and all have self-intersection -1. Hence, we can blow them down and obtain $\mathbb{P}^2$. It we blow up again the six points we obtain the same surface, but now three of the six exceptional lines corresponding to these points are elliptic and three are hyperbolic.

The proof of Theorem 6.2 given in [12] is very elegant but relies heavily on the geometry of the 27 lines and the fact that we are dealing with cubics. We are going to give another proof based on the construction of the mod 4 quadratic form associated with the embedding of the surface in $\mathbb{P}^3$.

7. EMBEDDED SURFACES IN $\mathbb{P}^3$

In the sequel, we will systematically use the fact that under Poincaré duality the mod 2 cup-product corresponds to the mod 2 intersection.

If $M = \mathbb{P}^3$ we also have the following well-known facts.

7.1. (i) If F is a non-orientable embedded surface the intersection of F with a generic plane is non-zero (if not it could be embedded in $\mathbb{R}^3$). Since the fundamental class of a generic plane generates $H_2(\mathbb{P}^3, \mathbb{Z}/2)$, the fundamental class of F is non-zero.

(ii) If we identify $\mathrm{H}_1(\mathbb{P}^3, \mathbb{Z}/2)$ with $\mathbb{Z}/2$ the self-intersection of an embedded surface F in $\mathbb{P}^3$ is equal to the mod 2 Euler characteristic of F. In particular, if F is orientable then its fundamental class is zero and if F is non-orientable its Euler characteristic is odd (by (i)).

In particular, 7.1 implies, that two embedded surfaces F_1 and F_2 represent the same class in $\mathrm{H}_2(\mathbb{P}^3, \mathbb{Z}/2) \cong \mathbb{Z}/2$ if, and only if, they are both orientable or both non-orientable. Applying Proposition 4.2 we find the following result.

PROPOSITION 7.2. *Two embedded surfaces $[f_1]$ and $[f_2]$ in $\mathbb{P}^3(\mathbb{R})$ of types F_1 and F_2, respectively, are cobordant if and only if F_1 and F_2 are either both orientable or both non-orientable.*

COROLLARY 7.3. *Let $f, g: F \to \mathbb{P}^3(\mathbb{R})$ be two embeddings. Let Θ be a Spin structure on $\mathbb{P}^3(\mathbb{R})$. Then $q_{f,\Theta}$ and $q_{g,\Theta}$ are isometric.*

Proof. By Proposition 7.2 the two embedded surfaces $[f]$ and $[g]$ are cobordant. Hence, $q_{f,\Theta}$ and $q_{g,\Theta}$ have same Arf–Brown invariant by Proposition 4.3. But by 1.5 this implies that the two forms are isometric. □

As in Section 3 we fix on $\mathbb{P}^3(\mathbb{R})$ the Spin structure Θ such that $q_{j,\Theta} = q_+$ (see Section 1), where j is the canonical embedding of $\mathbb{P}^2(\mathbb{R})$ in $\mathbb{P}^3(\mathbb{R})$.

COROLLARY 7.4. *Let $[f]$ be an embedded surface in $\mathbb{P}^3$ of type F. Let Θ be the Spin structure on $\mathbb{P}^3$ we have fixed above. Then $q_{f,\Theta}$ is isometric to*

$$gQ_1$$

(see Section 1 for notations) if F is orientable of genus g or isometric to

$$hQ_1 \perp q_+$$

if F is non-orientable of Euler characteristic $1 - 2h$ (recall that a non-orientable embedded surface in $\mathbb{P}^3$ always has odd Euler characteristic; see 7.1).

If we consider the other Spin structure on $\mathbb{P}^3$ we only need to replace q_+ by q_-.

Proof. By the same argument as in the proof of Corollary 7.3 all we need to prove is that for a given topological type F, we can find an embedding f such that $q_{f,\Theta}$ is of the desired form.

If F is orientable we can take the standard unknotted embedding of F in $\mathbb{R}^3 \subset \mathbb{P}^3$. If F is non-orientable of Euler characteristic $1 - 2h$ we can realize the embedding by taking the connected sum of a surface $F' \subset \mathbb{R}^3 \subset \mathbb{P}^3$ of genus h and the plane at infinity (this can easily be made into an embedding). Since connected sums correspond to orthogonal sums for quadratic forms, we obtain the desired form by taking the standard embedding for F'. □

8. THE RESULT OF SEGRE REVISITED

We fix on $\mathbb{P}^3(\mathbb{R})$ the same Spin structure as above. Let X be a real and smooth cubic surface in $\mathbb{P}^3$ and let D be a line in X. Orient $D(\mathbb{R})$ in some way. As we move along $D(\mathbb{R})$ the tangent plane $T_{X,x} (x \in D(\mathbb{R}))$ turns in some direction. Since $T_{X,x} = T_{X,y}$, for x and y in D, implies $x = y$ or $x = \sigma y$ (where σ is the involution defined by the pencil of conics; see Section 6), we note that the rotation of $T_{X,x}$ changes direction at a point x_0 if and only if x_0

is a fixed point of σ. Hence, the tangent plane takes a full half turn between x and σx if D is elliptic and none if D is hyperbolic (see Section 6).

From the remarks made in Section 3, it is easy to see that D is elliptic (resp. hyperbolic) if and only if $q_{f,\Theta}(d) = -1$ (resp. $q_{f,\Theta}(d) = +1$), where f is the embedding of X in $\mathbb{P}^3$.

From this remark we are going to prove that Theorem 6.2 of Segre is a consequence of Corollary 7.4.

We will need to recall some facts on the relations between $\mathrm{H}_2(X(\mathbb{C}), \mathbb{Z})$ and $\mathrm{H}_1(X(\mathbb{R}), \mathbb{Z}/2)$ for rational surfaces.

Let X be a complex rational surface (that is birationally equivalent to $\mathbb{P}^2(\mathbb{C})$—this is the case for cubic surfaces) defined over $\mathbb{R}$. For such surfaces we have a natural isomorphism (see [13, Ch. III]),

$$\mathrm{H}^1(G, \mathrm{H}_2(X(\mathbb{C}), \mathbb{Z})) \cong \mathrm{H}_1(X(\mathbb{R}), \mathbb{Z}/2),$$

where G is the Galois group $Gal(\mathbb{C} \mid \mathbb{R})$.

This isomorphism can be made quite explicit and described as follows. Recall that $\mathrm{H}^1(G, \mathrm{H}_2(X(\mathbb{C}), \mathbb{Z}))$ is defined as the quotient of the anti-invariant part of $\mathrm{H}_2(X(\mathbb{C}), \mathbb{Z})$ under the action of G, modulo the image of $(1 - S)$, where S is the generator of G. Denote by $\mathrm{H}_2(X(\mathbb{C}), \mathbb{Z})(1)^G$ the subgroup of anti-invariant elements of $\mathrm{H}_2(X(\mathbb{C}), \mathbb{Z})$ and let $\gamma \in \mathrm{H}_2(X(\mathbb{C}), \mathbb{Z})(1)^G$. Since X is rational and hence the homology of $X(\mathbb{C})$ is generated by algebraic cycles we can apply [13, Proposition I, (4.5)] (we assume $X(\mathbb{R}) \neq \emptyset$) and represent γ by an algebraic curve C defined over $\mathbb{R}$ (recall that since C is of real dimension 2, the restriction of complex conjugation to C reverses the orientation). The real part of C is of real dimension $\leqslant 1$. If $\dim C(\mathbb{R}) = 1$ then $C(\mathbb{R})$ defines a 1-cycle in $X(\mathbb{R})$. If $\dim C(\mathbb{R}) < 1$ or $C(\mathbb{R}) = \emptyset$ we associate with it the 0 1-cycle. By [13, Ch. III], the homology class of the 1-cycle thus defined only depends on the homology class γ. Hence, define $\varphi(\gamma)$ to be zero if $\dim C(\mathbb{R}) < 1$ and $\varphi(\gamma)$ to be the class of $C(\mathbb{R})$ if not. We have defined a morphism,

$$\varphi : \mathrm{H}_2(X(\mathbb{C}), \mathbb{Z})(1)^G \to \mathrm{H}_1(X(\mathbb{R}), \mathbb{Z}).$$

This morphism is onto and its kernel is precisely the image of $(1 - S)$ (see again [13, Ch. III]); hence, φ induces the desired isomorphism. This isomorphism has an additional property, namely if we consider the standard intersection form on $\mathrm{H}_1(X(\mathbb{R}), \mathbb{Z}/2)$ and on $\mathrm{H}^1(G, \mathrm{H}_2(X(\mathbb{C}), \mathbb{Z}))$ the bilinear form induced by the intersection form on $X(\mathbb{C})$, then φ is an isometry.

Let X be a cubic surface defined over $\mathbb{R}$ with 27 real lines. Let $e_1, \ldots, e_6, l$ be the basis of $\mathrm{H}_2(X(\mathbb{C}), \mathbb{Z})$ defined in Section 5. It is easily seen that in this case $\mathrm{H}_2(X(\mathbb{C}), \mathbb{Z})$ is anti-invariant and that $\varphi(e_1), \ldots, \varphi(e_2), \varphi(l)$ form a basis of $\mathrm{H}_1(X(\mathbb{R}), \mathbb{Z}/2)$. Now by Corollary 7.4 the mod 4 quadratic form $q_{f,\Theta}$ associated with the embedding of $X(\mathbb{R})$ in $\mathbb{P}^3(\mathbb{R})$ is isometric to $3Q_1 \perp q_+$. Using the relations of Remarks 1.3 and 1.5 and the fact that the self-intersection of the e_i's and of l is odd, we find that we must be, up to permutation of the e_i's, in one of the following cases:

8.1.

(i) $q_{f,\Theta}(\varphi(e_1)) = \cdots = q_{f,\Theta}(\varphi(e_6)) = q_{f,\Theta}(\varphi(l)) = -1.$

(ii) $q_{f,\Theta}(\varphi(e_1)) = \cdots = q_{f,\Theta}(\varphi(e_4)) = +1,$
$q_{f,\Theta}(\varphi(e_5)) = q_{f,\Theta}(\varphi(e_6)) = q_{f,\Theta}(\varphi(l)) = -1.$

(iii) $q_{f,\Theta}(\varphi(e_1)) = \cdots = q_{f,\Theta}(\varphi(e_3)) = -1,$
$q_{f,\Theta}(\varphi(e_4)) = \cdots = q_{f,\Theta}(\varphi(e_6)) = q_{f,\Theta}(\varphi(l)) = +1.$

In fact, all cases can occur (see below). From this and the description of the lines given in 5.2

it is quite easy to see that we have $q_{f,\Theta}(d) = -1$ for 12 lines and $q_{f,\Theta}(d) = 1$ for 15 lines. Explicitly, we find -1 for

— E_1 to E_6 and the six C_j's in case 8.1(i);
— E_5, E_6, C_5, C_6 and the eight D_{ij}'s for which i or $j = 5$ or 6 but $(i, j) \neq (5, 6)$, in case 8.1(ii);
— E_1 to E_3, C_4 to C_6, $D_{12}, D_{13}, D_{23}, D_{45}, D_{46}$ and D_{56}, in case 8.1(iii).

The same method applies in the case the surface has less than 27 lines. We do the case when the surface has 15 first. It is obtained by blowing up four real points in $\mathbb{P}^2$ and a pair of complex conjugate points. One can take $(\varphi(e_1), \ldots, (\varphi(e_4), (\varphi(l))$ (where the e_i's are the classes of the exceptional lines associated with the four real points) as a basis of $\mathrm{H}^1(X(\mathbb{R}), \mathbb{Z}/2)$. Then again, by Corollary 7.4 the quadratic form $q_{f,\Theta}$ must be isometric to $2Q_1 \perp q_+$. This means that, up to permutations of the e_i's, we have the following.

8.2.

(i) $q_{f,\Theta}(\varphi(e_1)) = q_{f,\Theta}(\varphi(e_2)) = -1,$

$q_{f,\Theta}(\varphi(e_3)) = q_{f,\Theta}(\varphi(e_4)) = q_{f,\Theta}(\varphi(l)) = +1,$

or

(ii) $q_{f,\Theta}(\varphi(e_1)) = q_{f,\Theta}(\varphi(e_2)) = q_{f,\Theta}(\varphi(e_3)) = +1,$

$q_{f,\Theta}(\varphi(e_4)) = q_{f,\Theta}(\varphi(l)) = -1.$

From this an easy computation shows that in both cases we have $q_{f,\Theta}(d) = -1$ for six lines and $+1$ for nine. Explicitly, we find -1 for $E_1, E_2, C_3, C_4, D_{12}$ and D_{34} in case (i) and for $E_4, C_4, D_{14}, D_{24}, D_{34}$ and D_{56} in case (ii) (recall that D_{56} corresponds to the line passing through the two complex conjugate points and that its homology class in $X(\mathbb{R})$ is $\varphi(l)$).

Applying the same type of argument to the other cases one finds, with notations similar to above,

— $q_{f,\Theta}(\varphi(e_1)) = q_{f,\Theta}(\varphi(e_2)) = +1$, $q_{f,\Theta}(\varphi(l)) = -1$ (or $q_{f,\Theta}(\varphi(e_1)) = -1$ and $q_{f,\Theta}(\varphi(e_2)) = q_{f,\Theta}(\varphi(l)) = +1$) in the case the surface has seven real lines,
— $q_{f,\Theta}(\varphi(l)) = +1$ in the two cases when the surface has three lines,

and it is again easy to find the result of Theorem 6.2 (note that in the last two cases all three lines have a non-trivial class; hence, the same class in $\mathrm{H}_1(X(\mathbb{R}), \mathbb{Z}/2) \cong \mathbb{Z}/2$).

We have said above that the three situations described in 8.1 can occur. To see this recall that any set of six non-intersecting lines in X can be blown down to give back $\mathbb{P}^2$. The different possibilities for $q_{f,\Theta}$ correspond exactly to base changes of $\mathrm{H}^1(X(\mathbb{R}), \mathbb{Z}/2)$ that replace the e_i's by an other set of non-intersecting six lines.

These three cases have an interesting interpretation in terms of the possible configurations of six points in the plane with respect to the conics passing through five of them. As described in Segre [12, p. 91], there are three possibilities. The first corresponds to the case when each of the six points lie inside the conic passing through the other five (recall that a conic in $\mathbb{P}^2(\mathbb{R})$ always has an inside and an outside, the outside being homeomorphic to a Möbius band), the second to the case when two are inside and four are outside and the last to the case when three are inside and three are outside. Recalling that the pencil of conics associated with the line that is the pullback of a conic through five points is the pullback of the pencil of lines through the sixth, it is easy to check that these three cases correspond to the three cases of 8.1.

The two forms of 8.2 can also be interpreted in terms of configurations of four points with respect to conics. Choosing the two complex points to be the cyclic points at infinity we can describe this in terms of circles passing through three points. In the first case, two of the real points are interior to the circle passing through the three others and two are exterior; in the second, three are exterior and one is interior.

The preceding discussion may lead to some confusion. It shows how the expression of the form $q_{f,\Theta}$ depends on the choice of six points in $\mathbb{P}^2$ but the form itself is of course the same. For cubic surfaces we have an even stronger result.

PROPOSITION 8.3. *For a real cubic surface X in $\mathbb{P}^3$, the* mod 4 *quadratic form associated with the embedding of $X(\mathbb{R})$ in $\mathbb{P}^3(\mathbb{R})$ is independent of the choice of the algebraic embedding.*

Proof. The result follows from the fact that the embedding of X in $\mathbb{P}^3$ itself *is canonical.* More precisely, it is associated with the anti-canonical class, and two embeddings of X in $\mathbb{P}^3$ differ by an automorphism of $\mathbb{P}^3$. Since an automorphism of $\mathbb{P}^3$ does not change the type, elliptic or hyperbolic, of a line in X we have the result. □

On the other hand, one should beware that for a general surface the forms depends on the embedding (but not its isometry class of course).

9. RELATIONS WITH THE QUADRATIC FORM OF ROHLIN

If a cubic surface X has 27 real lines then it is well known that X is an M-surface (that is $\sum \dim \mathrm{H}_i(X(\mathbb{C}), \mathbb{Z}/2) = \sum \dim \mathrm{H}_i(X(\mathbb{R}), \mathbb{Z}/2)$). For such surfaces, Guillou and Marin, generalizing results of Rohlin, have also constructed a quadratic form mod 4. That this is not the same as the quadratic form we have computed is easily seen by comparing the Arf–Brown invariants. In the case of the cubic with 27 real lines our form has Arf–Brown invariant equal to 1 (considered as an element of $\mathbb{Z}/8$) while the Rohlin form has invariant $-5 \equiv 3 \pmod 8$ (see [6, p. 98]).

In fact, we can describe the Rohlin form in the following way. Let $p_1, \ldots, p_6$ be the six points in general position in $\mathbb{P}^2(\mathbb{R})$. Let $E_1, \ldots, E_6$ be the corresponding real lines in $X(\mathbb{R})$ and $f_1, \ldots, f_6$ their classes in $\mathrm{H}_1(X(\mathbb{R}), \mathbb{Z}/2)$. Since blowing up is the same as taking a connected sum with $\mathbb{P}^2(\mathbb{C})$ with reversed orientation, we see that we have $q_{\mathbf{R}}(f_i) = -1$. On the other hand, if we let h be the class of the pullback in $X(\mathbb{R})$ of a general line in $\mathbb{P}^2(\mathbb{R})$ then $q_{\mathbf{R}}(h)$ is the same as for a line in $\mathbb{P}^2(\mathbb{R}) \subsetneq \mathbb{P}^2(\mathbb{C})$; hence equal to 1. An easy computation shows then, that we have $q_{\mathbf{R}}(d) = -1$ for all 27 lines in $X(\mathbb{R})$. One should note that the class h above is not canonical, i.e. depends on the choice of the six points (or if one prefers, the choice of the six non-intersecting lines), but it is easy to check that the description of $q_{\mathbf{R}}$ we have given is independent of this choice.

On the other hand, the description of $q_{f,\Theta}$ depends on this choice and we must make things independent. For this we first consider not h but the canonical class. To do this, let K be the canonical class on $X(\mathbb{C})$, then it is well known that for the canonical class (mod 2) k on $X(\mathbb{R})$ we have $k = \varphi(K)$ (see Section 8 for the definition of φ). We have $K = \sum e_i - 3l$ on $X(\mathbb{C})$ and hence $k = \varphi(l) + \sum \varphi(e_i) = h + \sum f_i$ on $X(\mathbb{R})$. Second we note that if $d_1, \ldots, d_6$ are the classes of six non-intersecting elliptic lines, that is $(d_i, d_j) = \delta_{ij}$ and $q_{f,\Theta}(d_i) = -1$, then $\sum d_i$ is independent of the choice of the d_i's. To see this recall that the 12 elliptic lines form a double six (see [12] or the computations made above) and that they split into exactly two subsets of six non-intersecting lines. From this, the assertion reduces to a trivial computation. We can now state the following proposition.

PROPOSITION 9.1. *Let X be a real cubic surface with 27 real lines. Let $q_{f,\Theta}$ be the quadratic form associated with the embedding of $X(\mathbb{R})$ in $\mathbb{P}^3(\mathbb{R})$ and let $q_{\mathbf{R}}$ be the Rohlin form. Let $d_1, \ldots, d_6$ be the classes of six non-intersecting lines such that $q_{f,\Theta}(d_i) = -1$ for each i and let k be the canonical class. Then*

$$q_{\mathbf{R}}(x) = q_{f,\Theta}(x) + 2\left\langle\left(k + \sum_{i=1}^{6} d_i\right), x\right\rangle$$

where $\langle , \rangle$ is the intersection form on $X(\mathbb{R})$.

Proof. The choice of the d_i's essentially means that we are in the situation described in 8.1(i). In this case the assertion of Proposition 9.1 is trivial to check.

Another case in which we can compare the two forms is the case of $X = \mathbb{P}^1 \times \mathbb{P}^1$ embedded in $\mathbb{P}^3$ as the surface defined by $x^2 + y^2 - z^2 - w^2 = 0$. Let e_1 (resp. e_2) be the class of $\mathbb{P}^1 \times \{y_0\}$ (resp. $\{x_0\} \times \mathbb{P}^1$) in $\mathrm{H}_1(X(\mathbb{R}), \mathbb{Z}/2)$ and choose $f: X \subsetneq \mathbb{P}^3$ to send $\mathbb{P}^1 \times \{y_0\}$ (resp. $\{x_0\} \times \mathbb{P}^1$) to the line defined by $y + z = 0$ and $x - w = 0$ (resp. $y - z = 0$ and $x - w = 0$). By the remarks made in Section 3 we see that $q_{f,\Theta}(e_1) = 2$, $q_{f,\Theta}(e_2) = 0$ and, of course $q_{f,\Theta}(e_1 + e_2) = 0$. On the other hand, for the Rohlin form we have $q_{\mathbf{R}}(e_1) = q_{\mathbf{R}}(e_2) = 0$ and $q_{\mathbf{R}}(e_1 + e_2) = 2$. In other words,

$$q_{\mathbf{R}}(x) = q_{f,\Theta}(x) + 2\langle e_2, x\rangle$$

in this case. □

Remark 9.2. The Rohlin form exists in a somewhat more general situation than just M-surfaces and exists for surfaces in which $X(\mathbb{R})$ is characteristic (see [6]). This is the case for the cubic surface with two real connected components. Such a surface is obtained by taking a rational ruled surface with real part homeomorphic to two spheres and blowing up a point on one of the spheres (see [13, VI (5.4.5)]). From this description we see that the Rohlin form on this surface is q_- and hence again different from our form.

10. IMMERSED AND EMBEDDED SURFACES IN A SPUN 3-MANIFOLD (CONTINUED)

In Section 4 we have introduced various equivalence relations between immersed surfaces, homotopy, regular homotopy, cobordism and the relation defined by the Arf–Brown invariant. To be able to compare these with more ease, we introduce some notations. Let $[f]$ and $[g]$ be two immersed surfaces in a spun 3-manifold (M, Θ). We will write $[f] \sim_\varepsilon [g]$, where $\varepsilon = h, r, c$ or q to mean that

(i) $[f]$ and $[g]$ are homotopic if $\varepsilon = h$;
(ii) $[f]$ and $[g]$ are regularly homotopic if $\varepsilon = r$;
(iii) $[f]$ and $[g]$ are cobordant if $\varepsilon = c$;
(iv) $q_{f,\Theta}$ and $q_{g,\Theta}$ are isometric if $\varepsilon = q$.

We will only use this last notation if $[f]$ and $[g]$ are of the same topological type (this is the only case where it makes sense). Note that if the two forms are isometric, then the isometry is realized by a diffeomorphism of the surface, since the intersection form is respected.

Since for the remainder of this section we will be working with a fixed spun 3-manifold (M, Θ), we will write q_f in place of $q_{f,\Theta}$. Also for the remainder of this section we will be working with a fixed surface F. In particular, all immersed surfaces will be of the same topological type.

We have shown above the following implications.

10.1.

$$[f] \sim_r [g] \;\Rightarrow\; \{[f] \sim_h [g] \text{ and } [f] \sim_c [g]\} \;\Rightarrow\; \{[f] \sim_h [g] \text{ and } [f] \sim_q [g]\}.$$

We are going to see that in general these implications *cannot be reversed.*

Let us fix, as usual, an homotopy class ξ of maps from F to M. If f and g are immersions belonging to ξ we know from [8] (see (3.4)), that there exists a circle C embedded in F such that g, up to regular homotopy, is obtained from f by adding a kink along C. Unfortunately, for an immersed surface $[f]$ the notion of adding a kink to $[f]$ along C is not well defined, since a diffeomorphism φ will change C. To circumvent this difficulty we proceed as follows. Let K be a curve, embedded in M and lying in $[f]$, i.e. in the image of f for any representative f (note that for any representative, $f_{|C}$ is an embedding). We will say that $[g]$ is obtained from $[f]$ by adding a kink along K if there exist representatives f and g such that g is obtained from f by adding a kink along $C = f^{-1}(K)$. Note that if g is obtained from f by adding a kink along C, then $g \circ \varphi$ is obtained from $f \circ \varphi$ by adding a kink along $\varphi^{-1}(C) = (f \circ \varphi)^{-1}(K)$.

We are going to analyse under which conditions on K we have $[f] \sim_\varepsilon [g]$.

Let f be an immersion and denote by γ the class of $C = f^{-1}(K)$ in $\mathrm{H}_1(F, \mathbb{Z}/2)$. From the definition of q_f, we see that, although γ depends on f, $q_f(\gamma)$ only depends on $[f]$ and not on the choice of the representative f. We will use the notation

$$q_{[f]}(K)$$

to denote this number.

PROPOSITION 10.2. *$[f] \sim_g [g]$ if and only if $q_{[f]}(K) = 0$.*

Proof. Let f and g be such that g is obtained from f by adding a kink along $C = f^{-1}(K)$. We have, by the proof of Theorem 3.4, that

$$q_g(x) = q_f(x) + 2(\gamma . x)$$

where γ is the class of C.

On the other hand,

$$q_f(\gamma + x) = q_f(x) + q_f(\gamma) + 2(\gamma . x).$$

Since, obviously,

$$\sum_x \mathrm{i}^{q_f(\gamma + x)} = \sum_x \mathrm{i}^{q_f(x)},$$

combining these two relations we find that

$$A(q_g) = (-\mathrm{i})^{q_f(\gamma)} A(q_f) = (-\mathrm{i})^{q[f](K)} A(q_f).$$

Hence the proposition by 1.5. □

LEMMA 10.3. *If $[f] \sim_c [g]$, then $q_{[f]}(K) = 0$ (K as above) and the class $\delta \in \mathrm{H}_1(M, \mathbb{Z}/2)$ of K in M is* 0.

Proof. We recall a notion due to Hass and Hughes [8]. Let Σ_f be the locus of non-injectivity of f. We may assume, staying in the same regular homotopy class that $f(\Sigma_f)$ is formed by double curves with eventually isolated triple points in the image of f (such an immersion is called generic). Call $\delta_{f,M} \in \mathrm{H}_1(M, \mathbb{Z}/2)$ the class of $f(\Sigma_f)$. This only depends on the class $[f]$ and is a regular homotopy invariant.

Now adding a kink along $f^{-1}(K)$ makes K a double curve in the image. Hence, we have

$$\delta_{g,M} = \delta_{f,M} + \delta.$$

On the other hand, if $[f]$ and $[g]$ are cobordant, with cobordism map $h: X \to M \times [0,1]$ we can define, in a way similar to the above, the locus of non-injectivity Σ_h of h and consider $h(\Sigma_h)$ as a 2-chain in $M \times [0,1]$. The boundary of this 2-chain is by construction $f(\Sigma_f) + g(\Sigma_g)$. Hence, $f(\Sigma_f)$ and $g(\Sigma_g)$ have same class in $\mathsf{H}_1(M \times [0,1], \mathbb{Z}/2) = \mathsf{H}_1(M, \mathbb{Z}/2)$ and $\delta = 0$. □

We note that we have in fact proved that if $[f] \sim_c [g]$, then $f(\Sigma_f)$ and $g(\Sigma_g)$ are homologous as boundaries of $h(\Sigma_h)$. For later use we state this formally.

LEMMA 10.4. *If $[f] \sim_c [g]$ then $\delta_{f,M} = \delta_{g,M}$.*

PROPOSITION 10.5. *Assume that $q_{[f]}(K) = 0$ and that K represents $1 \in \pi_1(M)$. Then $[f]$ and $[g]$ are regularly homotopic.*

COROLLARY 10.6. *If $\pi_1(M) \cong \mathsf{H}_1(M, \mathbb{Z}/2)$ then*

$$[f_1] \sim_r [f_2] \quad \Leftrightarrow \quad [f_1] \sim_h [f_2] \text{ and } [f_1] \sim_c [f_2].$$

Proof. By Lemma 10.3, $[f_1] \sim_c [f_2]$ implies that $\delta = 0$ (δ the class of K in $\mathsf{H}_1(M, \mathbb{Z}/2)$). Since $\pi_1(M) \cong \mathsf{H}_1(M, \mathbb{Z}/2)$, K represents $1 \in \pi_1(M)$ and we can apply Proposition 10.5. □

Proof of Proposition 10.5. As usual choose f and g such that g is obtained from f by adding a kink along $C = f^{-1}(K)$. Let φ be a Dehn twist along C and let γ be the class of C in $\mathsf{H}_1(F, \mathbb{Z}/2)$. Since $\varphi_*(\alpha) = \alpha + (\gamma.\alpha)\gamma$, it is easy to check that the hypothesis $q_f(\gamma) = 0$ implies

$$q_g = q_f \circ \varphi_*.$$

On the other hand, from the construction of q_f (see Section 3), it is easy to see that

$$q_{f \circ \varphi} = q_f \circ \varphi_*$$

and hence

$$q_g = q_{f \circ \varphi}.$$

Now, by Theorem 3.4, if g and $f \circ \varphi$ are homotopic they are regularly homotopic and so are $[f]$ and $[g]$. Hence, we only need to show that g and $f \circ \varphi$ are homotopic, or equivalently that f and $f \circ \varphi$ are homotopic. To prove this last statement, let T be a tubular neighbourhood of C in F and $h: T \to D$, D a disk in M, realizing the homotopy to a point of $f_{|C}$. Now f and $f \circ \varphi$ are homotopic to maps f' and f'' such that

— f' and f'' coincide outside T (since φ is a Dehn twist).
— $f'_{|T}$ and $f''_{|T}$ factor through h.

Since the disk is contractible, f' and f'' are homotopic relatively to $F \setminus T$ and this ends the proof. □

PROPOSITION 10.7. *$[f] \sim_c [g]$ if, and only if, $q_{[f]}(K) = 0$ and the class δ of K in $\mathsf{H}_1(M, \mathbb{Z}/2)$ is zero.*

Proof. The "only if" part has been proved in Lemma 10.3.

Let us prove the "if" part.

Let A be a collar of K in $G = f(F) \subset M$. Then A defines a framing of $N_{K|M}$ (take an inward pointing vector in A and then complete) and the hypothesis $q_{[f]}(K) = 0$ implies that this is an even framing.

By a result of Kneser (see, for example, the book [6, pp. 56–72]) the fact that $\delta = 0$ implies that K bounds a surface S (not necessarily orientable) embedded in M. By [10, Theorem 4.3], the framing defined by a collar of K in S defines again an even framing.

Since both framings are even we may assume, performing, if necessary, a *rolling-up* in a tubular neighbourhood U of K in M (see [2]) that S is transverse to G along K and meets G transversely outside of a tubular neighbourhood U.

Now consider a Morse function

$$r : S \to [0, 1]$$

having a unique maximum such that $r^{-1}(0) = K$.

Since r has a unique maximum we can find $\alpha \in [0, 1]$ such that α is not a critical value and $r^{-1}(\alpha) = \hat{K}$ is homotopic to zero in M.

Following the level lines of r between 0 and α we can deform (F, f, K) into $(\hat{F}, \hat{f}, \hat{K})$. We can even make this deformation transverse and this yields a cobordism between $f : F \to M$ and $\hat{f} : \hat{F} \to M$. We can do the same with g and find a cobordism between g and $\hat{g} : \hat{F} \to M$.

Between two consecutive critical values of r we can extend the isotopy between the level lines to an ambient isotopy of M. At a critical point the analysis is essentially local and we can follow the argument of Pinkall [11, pp. 432–433]) and conclude that $q_{\hat{f}}(\hat{K}) = q_f(K) = 0$ (in fact one can check this directly by noting that the typical situation is the one described in Fig. 3).

But now $\hat{g}$ is obtained from $\hat{f}$ by adding a kink along $\hat{K}$. By Proposition 10.5 this means that $\hat{f}$ is regular homotopic to $\hat{g}$ and hence cobordant. This proves the result. □

Examples 10.8. We are now going to give simple examples showing that the implications of 10.1 cannot be reversed in general (see also Section 12 for other examples).

Consider $M = S^1 \times T_2$ (where T_2 is a compact, connected surface of genus 2).

Let C be the circle shown in Fig. 4.

Fix a Spin structure Θ on M such that if f is the inclusion of the torus $S^1 \times C$, then $q_{f,\Theta}(\{x\} \times C) = 0$ and $q_{f,\Theta}(S^1 \times \{y\}) = 0$.

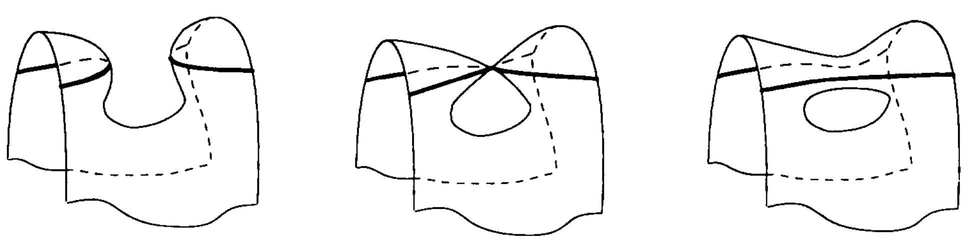

Fig. 3.

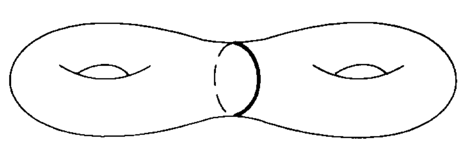

Fig. 4.

Let g_1 be obtained from f by adding a kink along $\{x\} \times C$. By Proposition 10.7 they are cobordant. On the other hand, since the mod 2 homology class of $\{x\} \times C$ in $S^1 \times C$ is not zero, g_1 and f are not regularly homotopic by Theorem 3.4. Now we note that since $f_*: \pi_1(S^1 \times C) \to \pi_1(M)$ is injective and $\pi_1(S^1 \times C)$ is abelian the only diffeomorphism φ of the torus such that $(f \circ \varphi)_* = f_*$ must be the identity. But g_1 and f are homotopic, hence $g_{1*} = f_*$ and $[f]$ and $[g_1]$ cannot be regularly homotopic.

Let g_2 be obtained from f by adding a kink along $S^1 \times \{y\}$. Then $[f] \sim_q [g_2]$ but they are not cobordant.

Propositions 10.2 and 10.5 suggest the following question. Assume that f and g are cobordant (hence g is obtained by adding a kink to f along a circle C such that $q_f(C) = 0$ and $[(C, f_{|C})] = 0 \in \mathrm{H}_1(M, \mathbb{Z}/2)$). If $f_*([C]) \neq 1$ in $\pi_1(M)$ when can we conclude that f and g are regularly homotopic?

One should expect that they are not in general regularly homotopic and this turns out to be the case, but there are exceptions. Before turning to these, however, we want to give some additional necessary conditions.

For simplicity, we will limit the following discussion to the cases of the *torus*. The general case can be treated along the same lines but is more involved and yields more cases.

We fix some notations. For the remainder of this section, F will denote a torus, M an orientable 3-manifold with a fixed Spin structure, $f: F \to M$ an immersion and C an oriented circle in F such that $f_{|C}$ is an embedding. We want to compare f and the immersion g obtained from f by adding a kink along C. By Proposition 10.2 we know that f and g will not be regularly homotopic if $q_f(C) \neq 0$. So we assume throughout that $q_f(C) = 0$.

Let α be the class of C in $\mathrm{H}_1(F, \mathbb{Z})$. By assumption on C we have $q_f(\alpha) = 0$ (where here we write α for its reduction mod 2) and we can find a basis (α, β) of $\mathrm{H}_1(F, \mathbb{Z})$ such that $q_f(\beta) = 0$ also.

If f and g are regularly homotopic, then there exists $\varphi \in \mathit{Diff}(F)$ such that

— $q_g = q_{f \circ \varphi}$;
—$f \circ \varphi$, g (and hence f) are homotopic.

On the other hand,

— $q_g = q_{f \circ \rho}$

where ρ is the Dehn twist along C, such that,

— $\rho_*(\alpha) = \alpha$;
— $\rho_*(\beta) = \alpha + \beta$.

Setting $\Omega = \varphi \circ \rho^{-1}$ we get

— $q_f = q_{f \circ \Omega}$;
— $f \circ \Omega$ and $f \circ \rho^{-1}$ are homotopic.

Let A be the matrix of

$$\Omega_*: \mathrm{H}_1(F, \mathbb{Z}) \to \mathrm{H}_1(F, \mathbb{Z})$$

with respect to the basis (α, β).

Since $q_f = q_{f \circ \Omega}$, we must have either

$$A \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod 2 \tag{10.9a}$$

or

$$A \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{mod } 2. \tag{10.9b}$$

Moreover, as $f \circ \Omega$ and $f \circ \rho^{-1}$ are homotopic, we have the following.

10.10
(i) $(f \circ \Omega)_*(\alpha) = f_*(\alpha)$;
(ii) $(f \circ \Omega)_*(\beta) = f_*(-\alpha + \beta)$.

Relations 10.9 and 10.10 show that, if f and g are regularly homotopic, $f_*(\alpha)$ and $f_*(\beta)$ must satisfy some non-trivial relations. In particular, if the subgroup G of $\mathrm{H}_1(M, \mathbb{Z})$ generated by the images of α and β is isomorphic to $\mathbb{Z}$, then taking reduction mod 2 shows that we must have $f_*(\alpha) \equiv 0$ mod 2 in G, identified with $\mathbb{Z}$. But in this case f and g can be regular homotopic as is shown by the following example.

Let, as before, F be a torus and (α, β) be a basis of $\mathrm{H}_1(F, \mathbb{Z})$. Let $M = S^1 \times T_2$, where T_2 is a surface of genus two. Let C be the curve indicated in Fig. 4, x_0 a point in S^1 and $\Gamma = \{x_0\} \times C$. The class of Γ in $\mathrm{H}_1(M, \mathbb{Z})$ is zero but its class in $\pi_1(M)$ generates an infinite cyclic subgroup. Let U be a tubular neighbourhood of Γ in M and let γ (resp. δ) be a longitude (resp. meridian) on $\partial U \cong S^1 \times S^1$.

Let $f: F \to \partial U$ be a diffeomorphism such that

$$f_*(\alpha) = 2\gamma + \delta, \qquad f_*(\beta) = 3\gamma + \delta$$

that we consider as an immersion of F in M. For a fixed Spin structure on M we can always choose γ and δ in M in such a way that $q_f(\alpha) = q_f(\beta) = 0$.

Now let g be obtained from f by adding a kink along a curve representing $f_*(\alpha)$. We have $g_g(\alpha) = 0$ and $q_g(\beta) = 2$. On the other hand, if $\psi \in \mathit{Diff}(F)$ is such that $\psi_*(\alpha) = \alpha$ and $\psi_*(\beta) = 3\alpha - \beta$, then f is homotopic to $f \circ \psi$ and $q_g = q_{f \circ \psi}$. By Theorem 3.4 we conclude that f and g are regular homotopic.

In the case G is finite we also have similar obstructions and in this case also, the surfaces can be regular homotopic when the obstructions are lifted. Here is an example. Let again F be a torus and let M be the lens space obtained by the surgery with coefficient 6 on the trivial knot in S^3. We have $\pi_1(M) \cong \mathbb{Z}/6$. Let K be a circle, embedded in M and generating $\pi_1(M)$. Let as above γ and δ be a longitude and a meridian on the boundary of a tubular neighbourhood of K and let f be an immersion associated with a diffeomorphism that maps α to $2\gamma + \delta$ and β to $3\gamma + \delta$. Letting $\psi \in \mathit{Diff}(F)$ be such that $\psi_*(\alpha) = \alpha$ and $\psi_*(\beta) = 3\alpha + \beta$, we can conclude as in the previous case.

11. THE COBORDISM GROUP OF IMMERSED SURFACES

The set of cobordism classes of immersed surfaces in a 3-manifold carries a natural semi-group structure induced by the action of taking disjoint unions. We will call this semi-group $N_2(M)$.

Remark 11.1. The action of taking immersed connected sums, denoted $\#$, is also well defined up to cobordism and in fact induces the same semi-group structure on $N_2(M)$. We will make frequent use of this fact by considering in many places connected sums in place of disjoint unions. On the other hand, one should beware that it is not well defined up to regular homotopy and not even up to homotopy, since there are infinitely many non-homotopic connected sums of parallel copies of F in $F + S^1$.

It is well known that $N_2(\mathbb{R}^3)$ is a group (the inverse is obtained by taking the reflection through a plane). As a by-product of our next result we are going to see that this is also the case for arbitrary orientable 3-manifolds M. But first let us fix some notations.

By Proposition 4.3, the Arf–Brown invariant of an immersed surface $[f]$ only depends on its cobordism class. Hence, we can speak of the Arf–Brown invariant A_X of an element X in $N_2(M)$. Pinkall [11] uses this construction to give a new proof of the result of Brown [4] that $N_2(\mathbb{R}^3) \cong \mathbb{Z}/8$. For commodity we will identify μ_8 with $\mathbb{Z}/8$ and consider A_X as an element of $\mathbb{Z}/8$.

Recall the notation $\delta_{f,M} \in \mathrm{H}_1(M, \mathbb{Z}/2)$ introduced in the proof of Lemma 10.3. By Lemma 10.4 $\delta_{f,M}$ is a cobordism invariant, and for $X \in N_2(M)$ we can define δ_X as $\delta_{f,M}$ for a representative f of X.

If $f: F \to M$ and $g: G \to M$ are cobordant, then (F, f) and (G, g) represent the same homology class in $\mathrm{H}_2(M, \mathbb{Z}/2)$ and we can speak of $H_X \in \mathrm{H}_2(M, \mathbb{Z}/2)$.

Finally, we will consider on

$$\Gamma(M) = \mathrm{H}_1(M, \mathbb{Z}/2) \times \mathrm{H}_2(M, \mathbb{Z}/2) \times \mathbb{Z}/8,$$

the following twisted group structure:

$$(\delta, H, A) + (\delta', H', A') = (\delta + \delta' + H.H', H + H', A + A')$$

where $H.H'$ is the intersection product in H_2.

THEOREM 11.2. *The map* $\psi: N_2(M) \to \Gamma(M)$ *defined by*

$$\psi(X) = (\delta_X, H_X, A_X)$$

is an isomorphism. In particular $N_2(M)$ *is a group.*

Proof. We first note that for $\mathbb{R}^3$ the map establishes an isomorphism $N_2(\mathbb{R}^3) \cong \mathbb{Z}/8$ (see [11]). This implies that $N_2(M)$ contains a subgroup $C(M)$, isomorphic to $\mathbb{Z}/8$, of immersed surfaces in a coordinate chart of M. Also, $N_2(M)$ contains the subset $E(M)$ of classes of embedded surfaces.

Let us prove that ψ is onto. Let $(\delta, H, A) \in \Gamma(M)$. Represent H by an element $F \in E(M)$. Represent δ by an embedded curve K in M. Consider a tubular neighbourhood of K in M and let T be its boundary torus. Add a kink along a longitude K' of T to obtain an immersed surface $[f: T \to M]$. By construction, $H_{[f]} = 0$ and hence the intersection $[f] \cdot F$ is zero. If $A_{[f]} + A_F = B$, take $S \in C(M)$ such that $A_S = A - B$. Clearly, since S lies in a coordinate chart, we also have $([f] + F).S = 0$. But this means that $\psi([f] + F + S) = (\delta, H, A)$ as desired.

We will need the following lemma.

LEMMA 11.3. *Every* $X \in N_2(M)$ *can be represented by a sum* $F + S + [f]$, $F \in E(M)$, $S \in C(M)$ *and* $[f]$ *as above. Moreover, we can choose the decomposition in such a way that* $q_{[f]}(K') = 0$ *hence so that* $A_{[f]} = 0$.

Let us show that the above lemma implies that ψ is injective. Let $X_1 = F_1 + S_1 + [f_1]$ and $X_2 = F_2 + S_2 + [f_2]$ be two decompositions as in Lemma 11.3 and assume that $\psi(X_1) = \psi(X_2)$. By construction $H_{X_i} = H_{F_i}$ so $H_{F_1} = H_{F_2}$ and F_1 and F_2 are cobordant by Proposition 4.2 and $A_{F_1} = A_{F_2}$ by Proposition 4.3. We also have by construction $A_{[f_i]} = 0$;

hence, $A_{S_1} = A_{S_2}$. Since S_1 and S_2 are in a chart the result of Pinkall mentioned above implies that S_1 and S_2 are cobordant.

Now, $[f_1] + [f_2]$ is cobordant to the connected sum $[f_1] \# [f_2]$ and this surface is obtained from the surface $T_1 \# T_2$ by adding kinks along disjoint longitudes K'_1 and K'_2 such that $q_{T_1 \# T_2}(K'_i) = q_{T_i}(K'_i) = 0$. Let γ be a simple curve in $T_1 \# T_2$ representing $K'_1 + K'_2$ mod 2. Then $[f_1] \# [f_2]$ is cobordant to the surface obtained from $T_1 \# T_2$ by adding a kink along γ. But now $\gamma = \delta_{[f_1]} + \delta_{[f_2]} = 0$ and $q_{T_1 \# T_2}(\gamma) = q_{T_1 \# T_2}(K'_1) + q_{T_1 \# T_2}(K'_2) = 0$. By Proposition 10.7 this implies that $[f_1] \# [f_2]$ is cobordant to the bounding surface $T_1 \# T_2$; hence to 0. Applying this to two copies of $[f_1]$ we see that $[f_1]$ is its own inverse. Then applying to $[f_1]$ and $[f_2]$ we conclude that $[f_1] = [f_2]$.

Proof of Lemma 11.3. Let X be a generic immersion. We will proceed in several steps.

Step 1: $X = X' + C$ where $C \in C(M)$ and X' has no triple points.

Let B and $\bar{B}$ be the two versions of the Boy surface in a coordinate chart of M. Let x be a triple point of X. By taking the connected sum of X and B or of X and $\bar{B}$ (depending on the nature of the triple point) we can eliminate by regular homotopy the triple point x and the triple point of B (or of $\bar{B}$). On the other hand, $B + \bar{B}$ is cobordant to 0; hence, $X = (X \# B) + \bar{B}$ (resp. $X = (X \# \bar{B}) + B$) and $(X \# B)$ (resp. $(X \# \bar{B})$) has one triple point less than X. Repeating the operation for each triple point we obtain the claim.

Let X be without triple points. Then, in particular, the locus of double points of X is the disjoint union of simple (i.e. non-self-intersecting) double curves. Let K be one of these. A tubular neighbourhood of K in X can be considered as bundle with fibre isomorphic to $\{(x, y) \mid xy = 0, x^2 + y^2 \leqslant 1\}$. We can count the number, mod 4, of quarter turns this configuration does when moving along K. Denote by $\ell(K)$ this number (as is easily seen this characterizes the bundle).

Step 2: Let K_1 and K_2 be two disjoint simple double curves in X. Then we can replace X by a surface X' where K_1 and K_2 are replaced by a single simple double curve K such that $\ell(K) = \ell(K_1) + \ell(K_2)$.

Consider the situation described in Fig. 5. Make two holes in each configuration (as indicated in Fig. 6), then add a handle connecting P_1 and P_3 and a second handle connecting P_2 and P_4 as indicated in Fig. 7.

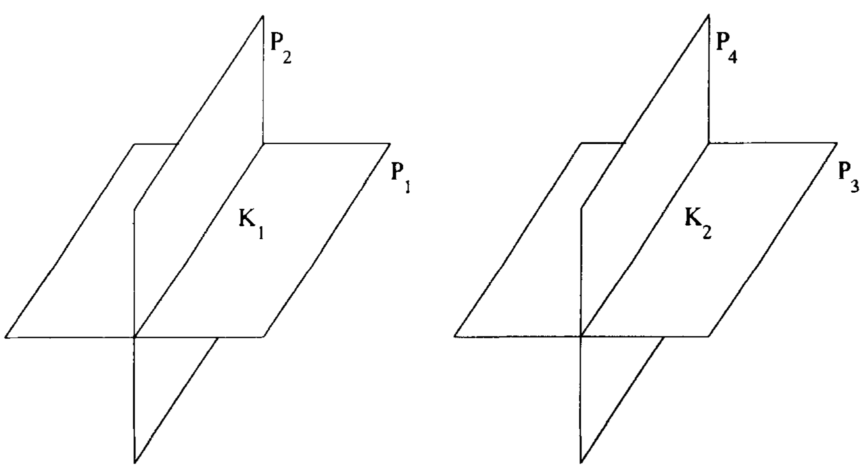

Fig. 5.

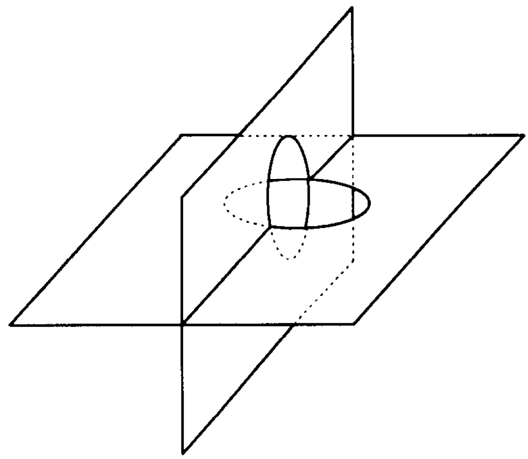

Fig. 6.

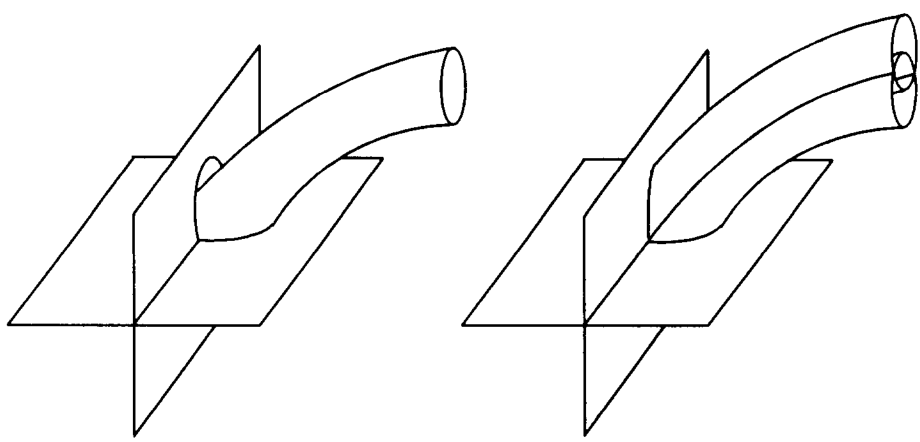

Fig. 7.

Note that the handle connecting P_1 and P_3 (resp. P_2 and P_4) does not intersect P_2 nor P_4 (resp. P_1 nor P_3) except at two points of K_1 and two points of K_2. Call Ha the union of these two handles. Adding extra handles if necessary and passing Ha "through" these handles, we may assume that Ha has no further intersections with X'. In particular, we have not added double curves or triple points.

This proves the claim.

Before starting our next step we need to introduce some notations.

Let F_1 be an immersion in a coordinate chart of M of the Klein bottle such that the locus of double points of F_1 is a simple double curve K_1, unknotted in M such that $\ell(K_1) = 2$. Let $\tilde{F}_1$ be an immersed surface in a chart of M such that $\tilde{F}_1 = F_1 + F$ and $\tilde{F}_1$ is cobordant to zero.

Let F_2 be the immersion of the torus in a chart of M obtained by adding a kink to a standard embedding T of the torus along a curve K_2 such that $q_T(K_2) = 2$. Let $\tilde{F}_2 = F_2 + F'$, $\tilde{F}_2$ cobordant to 0.

Step 3: Applying Step 1 we may assume that $X = X' + C$, $C \in C(M)$ and X' without triple points. Applying Step 2 we may assume that the locus of double points of X' is a simple and connected double curve K.

Assume that $\ell(K) = 2$. Then applying Step 2 to X' to $\tilde{F}_1$ we obtain $X = X'' + C'$, $C' \in C(M)$ and $\ell(K') = 0$, where K' is the double curve of X''.

Assume that $\ell(K) = 0$ but that $q_X(K) = 2$. Then applying again Step 2 to X' and $\tilde{F}_2$ we obtain $X = X'' + C'$, $C' \in C(M)$ and $l(K') = 0$ and $q_{X''}(K') = 0$, where K' is the double curve of X''.

Step 4: Let $X = X' + C$, $C \in C(M)$ and X' without triple points and locus of double points a simple connected curve K such that both $\ell(K)$ and $q_{X'}(K)$ are zero. Then $X = F + S + [f]$, F, S and $[f]$ satisfying conditions of Lemma 11.3.

For this we perform a "Rohlin surgery" along K which, in each fibre, makes the replacement shown in Fig. 8.

In such a way we isolate the double line K in a torus immersed in a tubular neighbourhood of the knot K, so that X' is replaced by the union of such an immersion and an embedded surface. Since $q_X(K) = 0$ this is again true in the new surface, hence the claim.

Combining steps 3 and 4 we see that the lemma is proved if $\ell(K) \equiv 0 \bmod 2$.

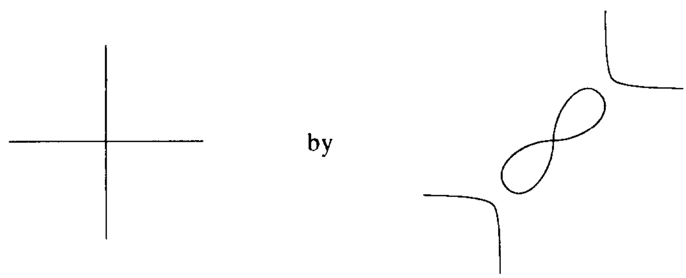

Fig. 8.

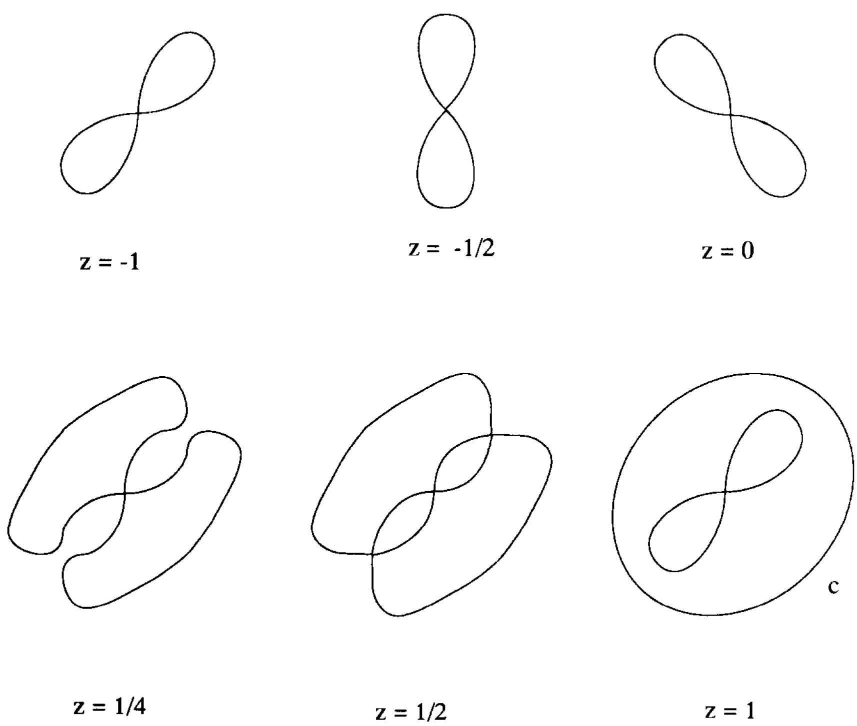

Fig. 9.

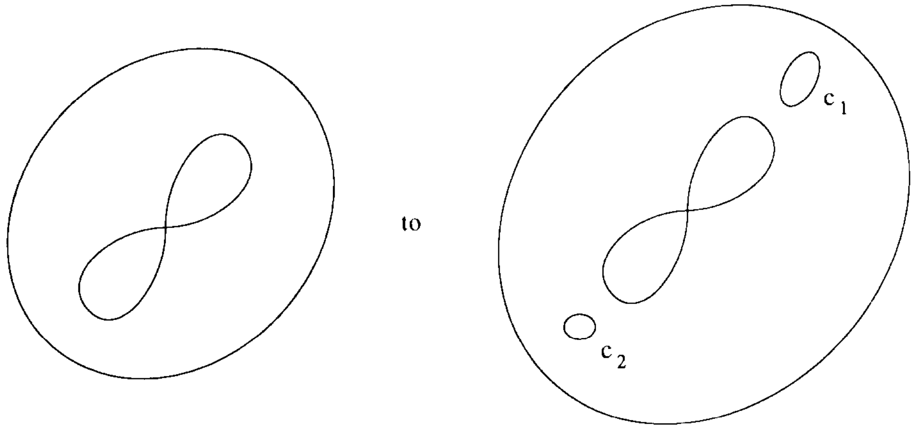

Fig. 10.

Step 5: It remains to consider the case $\ell(K) \equiv 1 \bmod 2$. For this we need to introduce the surfaces F_3 and $\bar{F}_3$. To construct these we consider the surface fibred over $-1 \leqslant z \leqslant 1$ with fibres as shown in Fig. 9.

We can complete this into an immersion in $\mathbb{R}^3$ of a surface of Euler characteristic -1 by bounding the simple curve c in the fibre above $z = 1$ by a disk D and identifying the two figure 8's of fibres above $z = -1$ and $z = 1$ (without twisting). This is F_3. We obtain $\bar{F}_3$ from F_3 by taking the mirror image of F_3, so that $F_3 + \bar{F}_3$ is cobordant to zero. Note that F_3 has a single triple point contained in the intersection of the simple double curve $K_3 = \{0\} \times [-1, 1]$ and the figure 8 contained in the disk D. Moreover $\ell(K_3) = 1$.

Let $X = X' + C$ as before and assume that for the simple connected double curve K of X' we have $\ell(K) = 1$. Applying Step 2 to X' and $\bar{F}_3$ we obtain $X = X'' + C'$, $C' \in C(M)$, X'' with a simple double curve K' such that $\ell(K') = 2$. On K' we have a single triple point situated as before in the intersection with the figure 8 in the disk D. Now we can perform a surgery similar to the one made in Step 4 and isolating the double curve in the immersion of a Klein bottle. Moreover, in the disk D we see the passage as shown in Fig. 10.

By further applications of Step 4 we can eliminate the double curves c_1 and c_2. Adding F_3 to this construction we have obtained $X = F + S + [h]$ where $[h]$ is an immersion of a Klein bottle in a tubular neighbourhood of K'. But in this case, although we still have a triple point, we can apply Step 3 and again Step 4 to obtain a decomposition satisfying the conditions of the lemma. If $\ell(K) = 3$ we can do the same construction replacing F_3 by $\bar{F}_3$. This proves Lemma 11.3 and that ψ is bijective. Since ψ is obviously a morphism this ends the proof of Theorem 11.2 □

Remark 11.4. The difference between the cases $\ell(K) \equiv 0 \bmod 2$ and $\ell(K) \equiv 1 \bmod 2$ is that in the decomposition $F + S + [f]$, $F + [f]$ has no triple point in the first case whereas $F + [f]$ carries one in the second. The difference comes from the mod 2 intersection number $(\delta_X . H_X)$ in M.

12. COMPLEMENTS ON IMMERSED SURFACES IN $\mathbb{P}^3$

We first note that in the case of $\mathbb{P}^3$ we have the following application of the results of Sections 4 and 10.

PROPOSITION 12.1. (i) *Two homotopic and cobordant immersed surfaces in* $\mathbb{P}^3$ *are regularly homotopic.*

(ii) *Two embedded surfaces in* $\mathbb{P}^3$ *are regularly homotopic if and only if they are homotopic.*

Proof. (i) is just a special case of Corollary 10.6 and by Proposition 4.2; (i) implies (ii). □

We also note that in the case of $\mathbb{P}^3$, Theorem 11.2 yields the following theorem.

THEOREM 12.2. $N_2(\mathbb{P}^3)$ *is isomorphic to* $\mathbb{Z}/4 \times \mathbb{Z}/8$.

All we need to note is that the twisted group structure is $\mathbb{Z}/4 \times \mathbb{Z}/8$.

As a last application of our methods we are going to show that one can select a finite set of *elementary* immersed surfaces in $\mathbb{P}^3$ such that every immersed surface can be realized as the connected sum of elementary ones, generalizing what is done in [11] for $\mathbb{R}^3$. Here is a list of elementary immersed surfaces in $\mathbb{P}^3$.

12.3.

(i) T is the standard embedded torus in an affine part of $\mathbb{P}^3$.
(ii) T' is the non-trivial immersion of the standard torus in an affine part of $\mathbb{P}^3$. This is obtained from T by adding a kink along $\alpha + \beta$, where $\{\alpha, \beta\}$ is a standard basis of $\mathrm{H}_1(T, \mathbb{Z}/2)$ such that $q(\alpha) = q(\beta) = 0$.
(iii) B and B' are the two Boy immersions of $\mathbb{P}^2$ in an affine part of $\mathbb{P}^3$.
(iv) T^* is an embedding of the standard torus and $f^*(h)$ is non-trivial (h being the generator of $\mathrm{H}^1(\mathbb{P}^3, \mathbb{Z}/2)$).
(v) $T^{*\prime}$ is obtained from T^* by adding a kink along $\alpha + \beta$, where $\{\alpha, \beta\}$ is again a standard basis of $\mathrm{H}_1(T^*, \mathbb{Z}/2)$ with $q(\alpha) = 0$ and $q(\beta) = 0$.
(vi) $T^{*\prime\prime}$ is obtained from T^* by adding a kink along α, $q(\alpha) = 0$ and α non-trivial in $\mathrm{H}_1(\mathbb{P}^3, \mathbb{Z}/2)$ (note that this means that β is dual to $f^*(h)$ in T^*).
(vii) B^* is the standard embedding of $\mathbb{P}^2$ in $\mathbb{P}^3$.
(viii) $B^{*\prime}$ is obtained from B^* by adding a kink along the non-trivial element of $\mathrm{H}_1(B^*, \mathbb{Z}/2)$.

PROPOSITION 12.4. *Every immersed surface in* $\mathbb{P}^3$ *is regular homotopic to an immersed connected sum of surfaces in the above list.*

Proof. First recall the well-known lemma.

LEMMA 12.5. (i) *Two maps* $f, g: F \to \mathbb{P}^3$ *are homotopic if and only if* $f^*(h) = g^*(h)$ *where* h *is the generator of* $\mathrm{H}^1(\mathbb{P}^3, \mathbb{Z}/2)$;

(ii) *For every* $c \in \mathrm{H}^1(F, \mathbb{Z}/2)$ *there exists an* f *such that* $f^*(h) = c$.

Proof. Here is a quick proof: $\mathrm{H}^1(F, \mathbb{Z}/2)$ classifies, up to isomorphism, the line bundles on F. In other words, there is a one to one correspondence between $\mathrm{H}^1(F, \mathbb{Z}/2)$ and $[F, B_{O(1)}]$, via $f \leftrightarrows f^*(\omega)$, ω being the first Stiefel–Whitney class of the universal line bundle. On the other hand, one can prove directly that every line bundle on F carries a classifying map $f: F \to \mathbb{P}^3$. If f and g are two such maps, any homotopy between f and g, eventually regarded as maps into $\mathbb{P}^n$, with n big enough, can be "projected" onto $\mathbb{P}^3$. Hence, the assertions by the naturality of the Stiefel–Whitney class. $\square$

As a first consequence of Lemma 12.5(i) we note that, in the case of $\mathbb{P}^3$ we do not have the problem mentioned in Remark 11.1, and we can speak of the homotopy class, and hence of the regular homotopy class, of an immersed connected sum.

Proof of Proposition 12.4 (*continued*). Let $f: F \to \mathbb{P}^3$ be an immersed surface.

First assume that $f^*(h) = 0$. This case is easy (and in fact contained in [11]). We note that the quadratic forms corresponding to T, T', B and B' are, respectively, Q_1, Q_2, q_+ and q_-. By considering a suitable immersed connected sum of these components we can find an immersion g of F with quadratic form q_g isometric to q_f. Let φ be a diffeomorphism of F such that $q_f \circ \varphi_* = q_g$. Since

$$\varphi^*(f^*(h)) = \varphi^*(0) = 0 = g^*(h),$$

$f \circ \varphi$ and g are homotopic and we can apply Theorem 3.4.

If $f^*(h) \neq 0$, let γ be dual to $f^*(h)$.

If $(\gamma.\gamma) = 1$ we have an orthogonal sum decomposition $\langle\gamma\rangle \perp H$ of $\mathsf{H}_1(F, \mathbb{Z}/2)$. To this decomposition corresponds a connected sum decomposition $\mathbb{P}^2 \# F'$ of F. Depending on whether $q_f(\gamma) = 1$ or -1 there also corresponds a decomposition $q_+ \perp q'$ or $q_- \perp q'$ of q_f.

Proceeding as above, by taking connected sums of components of the form T, T', B or B', we can find an immersion g' of F' such that $q_{g'}$ is isometric to q'. Let φ'_* be the corresponding isometry of $\mathsf{H}_1(F', \mathbb{Z}/2) = H$. We can extend this to an automorphism φ_* of $\mathsf{H}_1(F, \mathbb{Z}/2)$ by letting $\varphi_*(\gamma) = \gamma$. Since φ_* lifts to a diffeomorphism of F we find that $[f]$ is homotopic to $B^* \# [g']$ (resp. $B^{*\prime} \# [g']$) and hence regularly homotopic, since by construction $q_f \circ \varphi = q_+ \perp q'$ (resp. $q_- \perp q'$).

If $(\gamma.\gamma) = 0$ there are two possibilities, either there exists γ', such that $(\gamma.\gamma') = 1$, $(\gamma'.\gamma') = 0$ and an orthogonal sum decomposition,

$$\mathsf{H}_1(F, \mathbb{Z}/2) = \Gamma \perp H,$$

where Γ is the subspace generated by γ and γ', or there is an orthogonal sum decomposition,

$$\mathsf{H}_1(F, \mathbb{Z}/2) = E \perp H,$$

where the restriction of the intersection form to H is even, E is generated by $\{e_1, e_2\}$, $(e_i.e_j) = \delta_{ij}$ (the Kronecker symbol) and $\gamma = e_1 + e_2$.

The first corresponds to a connected sum decomposition $\{torus\} \# F'$ and the second to $\{Klein\ bottle\} \# F'$.

Replacing B^* and $B^{*\prime}$ by T^*, $T^{*\prime}$ or $T^{*\prime\prime}$ (resp. $B^* \# B^*$, $B^* \# B^{*\prime}$ or $B^{*\prime} \# B^{*\prime}$), depending on whether the restriction of q_f to Γ (resp. E) is Q_1 with $q_f(\gamma) = 0$, Q_2 or Q_1 with $q_f(\gamma) = 2$ (resp. $2q_+$, $q_+ \perp q_-$ or $2q_-$), we can proceed essentially in the same way as in the case $(\gamma.\gamma) = 1$. $\square$

Remarks and examples 12.6. (i) Clearly, the embedded surfaces T and T^* are cobordant but not homotopic.

(ii) The immersed surfaces $B^* \# B'$ and $B^{*\prime} \# B$ are homotopic and q-equivalent but not regularly homotopic. The reason for this is that, if we call f the first immersion and g the second, the automorphism such that $q_f \circ \varphi_* = q_g$ will exchange the standard generators of H_1. But since $f^*(h) = g^*(h)$, $f \circ \varphi$ and g will not be homotopic. In fact, $f \circ \varphi$ and g are not even cobordant, since $\delta_{f, \mathbb{P}^3} = 0$ while $\delta_{g, \mathbb{P}^3} \neq 0$ (see Lemma 10.4).

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